
Analytical Solutions for Scale and Time Dependent Solute Transport in Heterogeneous Porous Medium

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Abstract: Contaminated groundwater has been a serious problem across the world for many years as it has a bad impact on the quality of groundwater as well as on the environment. This study considers the solute transport problem in a heterogeneous porous medium with scale and time-dependent dispersion. The heterogeneity of porous media at the microscopic level facilitates dispersion, which affects groundwater flow patterns and solute distribution. For this work, the porous formation is assumed to be of semi-infinite length and of adsorbing nature. The key parameters such as dispersion coefficient and groundwater velocity are considered to be spatially and temporally dependent functions in degenerated forms. In addition, the first-order decay and zero-order production terms are also considered as time-dependent functions. Initially, it is assumed that the aquifer is uniformly polluted. Two different types of input sources namely uniform and varying nature are considered along the flow at one end in two separate cases, while concentration gradient, at non-source end boundary, is supposed to be zero. An analytical solution of the current boundary value problem is obtained using the Laplace Integral Transform Technique (LITT). The results obtained from the proposed problem are demonstrated graphically for a particular time functions in dispersion and groundwater velocity.

Keywords: Advection, Dispersion, Scale and Time Dependent, Groundwater Velocity, Porous Media

1. Introduction

Groundwater is one of the major resources of fresh water worldwide. It is especially important in arid and semi-arid regions where rainfall and surface water is scarce. Groundwater contamination can have a negative impact on the environment, the quality of the land, and human health. Rectification of these contaminated water reserves has been a major challenge for researchers around the world. Contaminants, entering groundwater at some site by natural or man-made means, increase the level of whole groundwater domain contamination due to the flow dynamics of the aquifer. Once groundwater gets contaminated, its complex composition makes it very difficult to assess and propose remedial measures to eliminate it. Typically, groundwater flow and

solute transport in unsaturated aquifers are caused by transient phenomena, making it a difficult problem. An efficacious mathematical model is needed to describe, predict and control the solute transport phenomena in an aquifer system.

The advection-dispersion equation (ADE) is one of the most appropriate and popular equation to describe the solute transport problem in an aquifer system. This equation (ADE) may derive using conservation of mass and Fick's law of diffusion. The velocity and direction of solute transport depends on an understanding of groundwater velocity that determines convective velocity and its value. In a heterogeneous porous medium, solutes move along different trajectories, which is variation in velocity and dispersion.

There are many analytical/numerical solutions of the advection-dispersion equation in the literature, but none are

universally applicable to all cases, although their importance is undeniable. Solutions of the advection-dispersion equation with constant dispersion do not provide a satisfactory description of the transport of solutes in soil and groundwater [1, 2]. Lynn W Gelhar et al concluded that dispersion is time-dependent and continues to increase until an asymptomatic value is reached [3]. Raja Ram Yadav et al developed an analytical solution describing the concentration distribution in a one-dimensional saturated aquifer system [4]. A number of studies (theoretical and experimental) show the transport of pollutants in most natural porous media is affected by spatial variability in physical and chemical properties [5, 6]. Theoretically, Oktay Guven et al also found that the dispersion in layered porous media is time-dependent [7]. Andrew D Barry and Garrison Sposito developed an analytical solution for the advection-dispersion equation with a temporally dependent dispersion coefficient in a semi-infinite porous domain [8]. William J Golz et al developed analytical solutions to the convection-diffusion equations to account for chemical transport affected by adsorption, decay, and first order production [9]. Stochastic and deterministic are generally two techniques to describe solute transport with depth-dependent reactions in porous media [10].

Analytical solutions to the one-dimensional advection-dispersion equation in porous media affected by space- and time-dependent dispersion and velocity are obtained by [11, 12, 13]. Creating mathematical models to analyze the movement of pollutants while taking into account chemical processes and hydraulic conductivity is a challenging task [14]. Literature review suggests that there are many analytical/numerical solutions for different situations, but none can represent all situations [13, 15, 6, 16]. To explain solute transport processes in natural or artificial aquifer systems, time-dependent input source solutions are very useful [17]. An analytical solution to the convective-diffuse solute transport problem with time-dependent boundary conditions is presented by [18]. Mritunjay Kumar Singh et al, also developed scale-dependent dispersion in heterogeneous porous media [19]. Anis Younes et al investigated the effect of travel distance on the dispersivity value in porous media and observed that the difference in travel distance was mainly controlled by transverse diffusion [20]. In general, porous structures are heterogeneous in nature, exhibiting significant changes in hydraulic properties that affect flow and transport phenomena [21].

The purpose of this study is to propose an analytical solution for conservative solute transport in time- and space-dependent flow in heterogeneously adsorbing porous media. The focus of this study is to obtain analytical solutions using different forms of dispersion and groundwater velocity at different time intervals. It is assumed that both of these key parameters (dispersion and velocity) are space and time-dependent. The spatial dependence occurs due to the heterogeneity of the medium, while the temporal dependence occurs due to the unsteadiness of the flow field. The transport process in the aquifer is mainly influenced by boundary conditions as well as other physical processes. Finally, solutions are developed for two different input sources such

as uniform and varying nature. Moreover, the solute transport process also involves first-order decay and zero-order production terms. Here, we describe one-dimensional analytical solutions of the advection-dispersion equation involving the injection of a solute in a semi-infinite horizontally long porous formation that is initially, uniformly distributed with solute.

2. Mathematical Formulation and Analytical Solutions

This study mainly focused on describing the time- and space-dependent transport of pollutants in a semi-infinite heterogeneous porous structure. On the basis of some physical mechanisms/concepts like conservation of mass and Fick's law in solute transport problems mathematically, the advection-dispersion equation (ADE), which is a second-order partial differential equation, is applicable. For the proposed problem, flow is considered along longitudinal direction. In the Cartesian coordinate system, the left boundary of the medium is fixed at the origin, and the flow direction is supposed to be from left to right. The governing one-dimensional advection-dispersion equation (ADE) may be written as [22]:

$$R \frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left\{ D(x,t) \frac{\partial c}{\partial x} \right\} - \mu(x,t) c + \gamma(x,t) \quad (1)$$

Where, $c [ML^{-3}]$ is the contaminant concentration at position $x [L]$ and time $t [T]$. $D(x,t) [L^2 T^{-1}]$, $U(x,t) [LT^{-1}]$, $\mu(x,t) [T^{-1}]$ and $\gamma(x,t) [ML^{-3} T^{-1}]$ are dispersion coefficient, groundwater velocity, first order decay and zero order production parameters along longitudinal direction, respectively. Groundwater pollution, or the concentration of pollutants, in an aquifer system, is assumed to be varying over time. In (1), the effect of molecular diffusion is excluded from the solute transport process as the mechanical dispersion dominates the hydrodynamic dispersion process. The adsorption gives rise to the significant phenomena of delay in the movement of the solute transport, which is included in (1) as retardation factor R is a dimensionless quantity. As in real life scenarios, dispersion varies in different ways at different time and one cannot expect the same pattern to follow over time. With this view, the dispersion and velocity parameters are taken differently for different time slots. The space- and time-dependent expressions for the dispersion coefficient and groundwater velocity are assumed to be in the degenerate form as follows:

$$U(x,t) = U_0 (1+ax) F(t); D(x,t) = D_0 (1+ax)^2 F(t); \\ \mu = \mu_0 F(t); \gamma = \gamma_0 F(t) \quad (2)$$

where,

$$F(t) = \left\{ \begin{array}{l} f_1(m_1t); 0 < t \leq t_1 \\ f_2(m_2t); t_1 < t \leq t_2 \\ \dots\dots\dots \\ f_{n-1}(m_{n-1}t); t_{n-2} < t \leq t_{n-1} \\ f_n(m_nt); t_{n-1} < t \end{array} \right\}$$

It is to be noticed that the time function $F(t)$ is taken differently in different time domains. The expression $f_i(m_i t); i = 1, 2, \dots, n - 1, n$ is dimensionless and m_i is an

$$R \frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left\{ D_0 (1 + ax)^2 F(t) \frac{\partial c}{\partial x} \right\} - \frac{\partial}{\partial x} \left\{ U_0 (1 + ax) F(t) c \right\} - \mu_0 F(t) c + \gamma_0 F(t) \tag{3}$$

The solute transport is affected by the chosen boundary conditions. Analytical solutions are developed for uniform and varying nature inputs in the following two cases, respectively.

For the both cases, initial concentration is present in the background of the domain. Initially, the aquifer domain is assumed to be uniformly polluted, and the concentration gradient is zero at the other end of the boundary.

2.1. Case I: Uniform Input Point Source Condition

The introduction of solutes into a medium over time is mathematically represented by Dirichlet boundary conditions.

2.1.1. Initial and Boundary Conditions

It is assumed that some uniformly distributed concentration already exists at all points in the domain at the beginning of the investigation, i.e., at $t = 0$. The second type boundary condition with zero concentration gradient is imposed at the right end of the domain. At the one end, concentration introduced to the medium notices sudden jump for very short period of time, otherwise, it remains constant. To obtain the solution of (3), one initial condition and two boundary conditions are required, which can be expressed mathematically as follows:

$$c(x, t) = c_i; 0 \leq x < \infty, t = 0 \tag{4}$$

$$T = \int_0^t F(t) dt = \int_0^{t_1} f_1(m_1t) dt + \int_{t_1}^{t_2} f_2(m_2t) dt + \int_{t_2}^{t_3} f_3(m_3t) dt + \dots\dots\dots + \int_{t_{n-1}}^t f_n(m_nt) dt \tag{7}$$

Clearly, for $t = 0$, it takes $T = 0$ and for $t > 0$, T is also greater than zero. i.e., $T > 0$. Incorporating the transformation given with (7) in (3-6), all equations reduce to new form as follows:

$$R \frac{\partial c}{\partial T} = \frac{\partial}{\partial x} \left\{ D_0 (1 + ax)^2 \frac{\partial c}{\partial x} \right\} - \frac{\partial}{\partial x} \left\{ U_0 (1 + ax) c \right\} - \mu_0 c + \gamma_0 \tag{8}$$

$$c(x, T) = c_i; 0 \leq x < \infty, T = 0 \tag{9}$$

unsteady parameter having dimension inverse that of time. Here, for $f_1(m_1t)$ is chosen in such a way that $f_1(m_1t) = 1$ for $m_1 = 0$ or $t = 0$.

It is clear that the function $F(t)$ is dimensionless. D_0, U_0, μ_0 and γ_0 are the initial values of corresponding parameters at $x = 0$ and $t = 0$, and having the dimensions of corresponding parameters. Furthermore, a is a heterogeneity parameter whose dimension is inverse of that of position [13].

Incorporating, (2) into (1), we get as follows:

$$c(x, t) = c_0 + \psi \left[\begin{array}{l} u(t - t^*) \\ -u \left\{ t - (t^* + \mathcal{E}) \right\} \end{array} \right]; x = 0, t > 0 \tag{5}$$

$$\frac{\partial c(x, t)}{\partial x} = 0; x \rightarrow \infty, t \geq 0 \tag{6}$$

where, c_0 and c_i denote the reference and initial concentration, respectively and $u(t)$ is the unit step function of time variable and is considered dimensionless. Second term in right hand side of (5) represents input source like the waste material like toxic chemicals of amount $\psi [ML^{-3}]$ released suddenly at a time t^* and it lasts for very small time duration \mathcal{E} by the industry or the chemical compound seepage by the soil to the groundwater which used for the agriculture, and also during a volcanic eruption, lava, tephra (i.e., ash, lapilli). So, ψ may be termed as regulating coefficient. Groundwater contamination may occur due to sudden leakage (natural or unnatural activity) in an area relatively close to the inlet boundary.

2.1.2. Analytical Solution of the Problem

Now, a new time variable T is introduced with a transformation as follows [23]:

$$c(x, T) = c_0 + \psi \left[u \left(T - T^* \right) - u \left(T - T_{\epsilon}^* \right) \right]; x = 0, T > 0 \tag{10}$$

$$\frac{\partial c(x, T)}{\partial x} = 0; x \rightarrow \infty, T \geq 0 \tag{11}$$

It is obvious that $u(t - t^*)$ and $u\{t - (t^* + \epsilon)\}$ may be replaced by $u(T - T^*)$ and $u(T - T_{\epsilon}^*)$ respectively, because for $t = 0$, we have $T = 0$ and corresponding $t > 0$, we have $T > 0$, also for t^* and $t^* + \epsilon$, the corresponding time in new variables given in transformation in (7) are T^* and T_{ϵ}^* given as follows:

$$T^* = \int_0^{t^*} F(t) dt \text{ and } T_{\epsilon}^* = \int_0^{t^* + \epsilon} F(t) dt$$

To remove the spatial coefficients from (8), we further introduce another new transformation as follows [13]:

$$X = \int_0^x \frac{1}{(1+ax)} dx \text{ or } X = \frac{1}{a} \log(1+ax) \tag{12}$$

Obviously, the value of X is 0 for $x = 0$ and $X > 0$ for $x > 0$, using the transformation given in (12), (8-11) convert into following form:

$$R \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - (U_0 - D_0 a) \frac{\partial c}{\partial X} - (U_0 a + \mu_0) c + \gamma_0$$

or

$$R \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - U_1 \frac{\partial c}{\partial X} - \mu_1 c + \gamma_0 \tag{13}$$

where,

$$U_1 = U_0 - D_0 a, \mu_1 = U_0 a + \mu_0, c(X, T) = c_i; 0 \leq X < \infty, T = 0 \tag{14}$$

$$c(X, T) = c_0 + \psi \left[\begin{matrix} u \left(T - T^* \right) \\ -u \left(T - T_{\epsilon}^* \right) \end{matrix} \right]; X = 0, T > 0 \tag{15}$$

$$\frac{\partial c(X, T)}{\partial X} = 0; X \rightarrow \infty, T \geq 0 \tag{16}$$

To remove the convective term from (13), we further introduce a new transformation as follows:

$$c(X, T) = k(X, T) \exp \left[\begin{matrix} \frac{U_1}{2D_0} X - \\ \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1 \right) T \end{matrix} \right] + \frac{\gamma_0}{\mu_1} \tag{17}$$

Where, $k(X, T)$ is the new dependent variable, depending on the position and time. Using transformation (17), (13-16) reduces into following form:

$$R \frac{\partial k}{\partial T} = D_0 \frac{\partial^2 k}{\partial X^2} \quad (18)$$

$$k(X, T) = \left(c_i - \frac{\gamma_0}{\mu_1} \right) \exp(-\beta X); T = 0, X \geq 0 \quad (19)$$

$$k(X, T) = \left[c_0 + \psi \begin{bmatrix} u(T - T^*) \\ -u(T - T_{\varepsilon}^*) \end{bmatrix} - \frac{\gamma_0}{\mu_1} \right] \exp(\eta^2 T); X = 0, T > 0 \quad (20)$$

$$\frac{\partial k(X, T)}{\partial X} + \frac{U_1}{2D_0} k(X, T) = 0; X \rightarrow \infty, T > 0 \quad (21)$$

where,

$$\eta = \sqrt{\frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1 \right)}, \quad \beta = \frac{U_1}{2D_0}$$

Applying Laplace Integral Transformation Technique to (18-21), we have

$$\begin{aligned} \bar{k}(X, p) = & \left(c_0 - \frac{\gamma_0}{\mu_1} \right) \frac{1}{(p - \eta^2)} \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) + \frac{\psi}{(p - \eta^2)} \left[\exp\{-(p - \eta^2)T^*\} - \right. \\ & \left. \exp\{-(p - \eta^2)T_{\varepsilon}^*\} \right] \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) - \left(c_i - \frac{\gamma_0}{\mu_1} \right) \frac{1}{(p - \rho^2)} \times \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) + \left(c_i - \frac{\gamma_0}{\mu_1} \right) \frac{\exp(-\beta X)}{(p - \rho^2)} \end{aligned} \quad (22)$$

where,

$$\rho = \sqrt{\frac{D_0}{R} \beta^2}$$

Now taking Inverse Laplace Integral Transform of (22) and using transformation in (17) backward, the analytical solution of advection dispersion equation may be written as [24]:

$$\begin{aligned} c(X, T) = & \left\{ \left(c_0 - \frac{\gamma_0}{\mu_1} \right) G_{1\eta}(X, T) - \left(c_i - \frac{\gamma_0}{\mu_1} \right) G_{1\rho}(X, T) + \left(c_i - \frac{\gamma_0}{\mu_1} \right) \right. \\ & \left. \exp(-\beta X + \rho^2 T) \right\} \\ & \times \exp\left\{ \frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1 \right) T \right\} + \frac{\gamma_0}{\mu_1}, 0 < T \leq T^* \end{aligned} \quad (23)$$

$$c(X, T) = \left\{ \left(c_0 - \frac{\gamma_0}{\mu_1} \right) G_{1\eta}(X, T) + \psi \left\{ \frac{\exp(\eta^2 T^*)}{G_{1\eta}(X, T - T^*)} \right\} - \left(c_i - \frac{\gamma_0}{\mu_1} \right) G_{1\rho}(X, T) + \left(c_i - \frac{\gamma_0}{\mu_1} \right) \exp(-\beta X + \rho^2 T) \right\} \times \exp \left\{ \frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1 \right) T \right\} + \frac{\gamma_0}{\mu_1}, T^* < T \leq T_{\epsilon}^* \tag{24}$$

$$c(X, T) = \left\{ \left(c_0 - \frac{\gamma_0}{\mu_1} \right) G_{1\eta}(X, T) + \psi \left\{ \frac{\exp(\eta^2 T^*)}{G_{1\eta}(X, T - T^*)} - \exp(\eta^2 T_{\epsilon}^*) \times G_{1\eta}(X, T - T_{\epsilon}^*) \right\} - \left(c_i - \frac{\gamma_0}{\mu_1} \right) G_{1\rho}(X, T) + \left(c_i - \frac{\gamma_0}{\mu_1} \right) \exp(-\beta X + \rho^2 T) \right\} \times \exp \left\{ \frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1 \right) T \right\} + \frac{\gamma_0}{\mu_1}, T_{\epsilon}^* < T \tag{25}$$

where,

$$G_{1\eta}(X, T) = \frac{1}{2} \left\{ \exp \left(\eta^2 T - \eta \sqrt{\frac{R}{D_0}} X \right) \operatorname{erfc} \left(\frac{1}{2} \sqrt{\frac{R}{D_0 T}} X - \eta \sqrt{T} \right) + \exp \left(\eta^2 T + \eta \sqrt{\frac{R}{D_0}} X \right) \times \operatorname{erfc} \left(\frac{1}{2} \sqrt{\frac{R}{D_0 T}} X + \eta \sqrt{T} \right) \right\}$$

and $G_{1\rho}(X, T)$ is obtained by replacing η with ρ in $G_{1\eta}(X, T)$, while $G_{1\eta}(X, T - T^*)$ and $G_{1\eta}(X, T - T_{\epsilon}^*)$ are obtained by replacing T with $T - T^*$ and $T - T_{\epsilon}^*$, in $G_{1\eta}(X, T)$, respectively.

2.2. Case II: Varying Input Point Source Condition

This input is represented with robin type boundary conditions which is mixed type of Dirichlet’s boundary condition.

2.2.1. Initial and Boundary Conditions

Initial concentration distribution and boundary condition at infinity is same as taken in previous case expressed by (4) and (6), respectively. The input at $x = 0$ is assumed to be different from the previous case, which is mathematically expressed as:

$$-D(x, t) \frac{\partial c(x, t)}{\partial x} + U(x, t) c = U_0 \times \left[c_0 + \psi \left[u(t - t^*) - u \left\{ t - (t^* + \epsilon) \right\} \right] \right]; x = 0, t > 0 \tag{26}$$

2.2.2. Analytical Solution of the Problem

Using transformation given in (7) and incorporating (2), (26) reduces into following form:

$$-D_0 (1 + ax) \frac{\partial c(x, T)}{\partial x} + U_0 c = U_0 \left[c_0 + \psi \left[u(T - T^*) - u \left\{ T - (T^* + \epsilon) \right\} \right] \right]; x = 0, T > 0 \tag{27}$$

Using the transformation given in (12), (27) is converted into the following form:

$$-D_0 \frac{\partial c(X, T)}{\partial X} + U_0 c = U_0 \left[c_0 + \psi \left[u(T - T^*) - u(T - T_{\epsilon}^*) \right] \right]; X = 0, T > 0 \tag{28}$$

Using the transform given in (17) on (28), then the Laplace Integral Transform Technique on obtained equation along with (18, 19, 21) as performed in previous case, we have the following result:

$$\bar{k}(X, p) = \frac{U_0 c_0}{\sqrt{RD_0}} \frac{1}{(p - \eta^2)(\sqrt{p} + \sigma)} \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) + \frac{U_0 \psi}{\sqrt{RD_0}} \frac{1}{(p - \eta^2)(\sqrt{p} + \sigma)} \times$$

$$\left\{ \begin{array}{l} \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) \\ \exp\left\{-(p - \eta^2)T^*\right\} - \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) \exp\left\{-(p - \eta^2)T_{\mathcal{E}}^*\right\} \end{array} \right\} - \frac{U_0 \gamma_0}{\sqrt{RD_0} \mu_1} \frac{1}{(p - \eta^2)(\sqrt{p} + \sigma)} \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) \quad (29)$$

$$- \frac{U_0}{\sqrt{RD_0}} \left(c_i - \frac{\gamma_0}{\mu_1}\right) \frac{1}{(p - \rho^2)(\sqrt{p} + \sigma)} \exp\left(-\sqrt{\frac{pR}{D_0}} X\right) + \left(c_i - \frac{\gamma_0}{\mu_1}\right) \frac{\exp(-\beta X)}{(p - \rho^2)}$$

$$\text{Where, } \rho = \sqrt{\frac{D_0}{R} \beta^2}, \quad \sigma = \frac{2U_0 - U_1}{2\sqrt{RD_0}}$$

Now taking Inverse Laplace Integral Transform to (29) and using transformation (17) backward, the analytical solution of advection dispersion equation may be written as follows:

$$c(X, T) = \left\{ \frac{U_0 c_0}{\sqrt{RD_0}} G_{2\eta\sigma}(X, T) - \frac{U_0 \gamma_0}{\sqrt{RD_0} \mu_1} G_{2\eta\sigma}(X, T) - \frac{U_0}{\sqrt{RD_0}} \left(c_i - \frac{\gamma_0}{\mu_1}\right) G_{2\rho\eta}(X, T) + \right.$$

$$\left. \left(c_i - \frac{\gamma_0}{\mu_1}\right) \exp(-\beta X + \rho^2 T) \right\} \times \exp\left[\frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1\right) T \right] + \frac{\gamma_0}{\mu_1}, 0 < T \leq T^* \quad (30)$$

$$c(X, T) = \left\{ \frac{U_0 c_0}{\sqrt{RD_0}} G_{2\eta\sigma}(X, T) + \frac{U_0 \psi}{\sqrt{RD_0}} G_{2\eta\sigma}(X, T - T^*) \exp(\eta^2 T^*) - \frac{U_0 \gamma_0}{\sqrt{RD_0} \mu_1} \right.$$

$$G_{2\eta\sigma}(X, T) - \frac{U_0}{\sqrt{RD_0}} \left(c_i - \frac{\gamma_0}{\mu_1}\right) G_{2\rho\sigma}(X, T) + \left(c_i - \frac{\gamma_0}{\mu_1}\right) \exp(-\beta X + \rho^2 T) \left. \right\} \times$$

$$\exp\left[\frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1\right) T \right] + \frac{\gamma_0}{\mu_1}, T^* < T \leq T_{\mathcal{E}}^* \quad (31)$$

$$c(X, T) = \left\{ \frac{U_0 c_0}{\sqrt{RD_0}} G_{2\eta\sigma}(X, T) + \frac{U_0 \psi}{\sqrt{RD_0}} \right.$$

$$\left[\frac{G_{2\eta\sigma}(X, T - T^*) \exp(\eta^2 T^*)}{G_{2\eta\sigma}(X, T - T_{\mathcal{E}}^*) \exp(\eta^2 T_{\mathcal{E}}^*)} - \frac{U_0 \gamma_0}{\mu_1 \sqrt{RD_0}} G_{2\eta\sigma}(X, T) - \frac{U_0}{\sqrt{RD_0}} \right.$$

$$\left. \left(c_i - \frac{\gamma_0}{\mu_1}\right) G_{2\rho\sigma}(X, T) + \left(c_i - \frac{\gamma_0}{\mu_1}\right) \exp(-\beta X + \rho^2 T) \right\} \times \exp\left[\frac{U_1}{2D_0} X - \frac{1}{R} \left(\frac{U_1^2}{4D_0} + \mu_1\right) T \right] + \frac{\gamma_0}{\mu_1}, T_{\mathcal{E}}^* < T$$

$$(32)$$

where,

$$G_{2\eta\sigma}(X, T) = \frac{1}{2(\eta + \sigma)} \exp\left(\eta^2 T - \eta \sqrt{\frac{R}{D_0}} X\right) \operatorname{erfc}\left(\frac{\sqrt{R}}{2\sqrt{D_0} T} X - \eta \sqrt{T}\right) - \frac{1}{2(\eta - \sigma)} \times$$

$$\exp\left(\eta^2 T + \eta \sqrt{\frac{R}{D_0}} X\right) \operatorname{erfc}\left(\frac{\sqrt{R}}{2\sqrt{D_0} T} X + \eta \sqrt{T}\right) + \frac{\eta}{\eta^2 - \sigma^2} \exp\left(\sigma^2 T + \sigma \sqrt{\frac{R}{D_0}} X\right) \times \operatorname{erfc}\left\{\frac{\sqrt{R}}{2\sqrt{D_0} T} X + \sigma \sqrt{T}\right\}$$

and $G_{2\rho\sigma}(X, T)$ is obtained by replacing η with ρ in

$G_{2\eta\sigma}(X, T)$ while $G_{2\eta\sigma}(X, T - T^*)$ and $G_{2\eta\sigma}(X, T - T_{\epsilon}^*)$ are obtained by replacing T with $T - T^*$ and $T - T_{\epsilon}^*$, respectively in $G_{2\eta\sigma}(X, T)$ respectively.

3. Results and Discussion

Analytical solutions (23, 24, 25) and (30, 31, 32) to the governing equation are demonstrated graphically in this section which is based on published and experimental hydrological input data. The values of the parameters and empirical constants involved in the boundary conditions, as well as the governing equation, are determined from published papers like [12]. The parameters affecting the concentration distribution are demonstrated with help of various graphs. The contaminant concentration values are evaluated for the longitudinal porous domain $0 \leq x(km) \leq 4$. The values of common input parameters and constants are considered as follows:

$$c_0 = 1.0, \quad c_i = 0.1, \quad U_0 (\text{km/year}) = 0.65, \quad \mu_0 (\text{year}^{-1}) = 0.11, \quad \gamma_0 = 0.11, \quad \epsilon (\text{year}) = 0.05, \quad a (\text{km}^{-1}) = 0.1, \quad \psi (\text{year}) = 0.5, \quad t^* (\text{year}) = 0.80.$$

The mathematical expression for the time-dependent function that varies in each time domain is as follows:

$$F(mt) = \begin{cases} \exp(m_1 t); & 0 < t(\text{year}) \leq 1 \\ s_1 \exp\{m_2(t-1)\} + s_2; & 1 < t(\text{year}) \leq 5 \\ s_3 \frac{m_3(t-5)}{m_3(t-5)+1} + s_4; & t(\text{year}) > 5 \end{cases} \quad (33)$$

The values of the unsteady parameters are considered as follows:

$m_1 = 0.1(\text{year}^{-1}), m_2 = 0.3(\text{year}^{-1})$ and $m_3 = 0.4(\text{year}^{-1})$ also the time t_1 and t_2 are taken $1(\text{year})$ and $5(\text{year})$, respectively and the value of $s_1 = 0.36839, s_2 = 0.736781, s_3 = 0.917324$ and $s_4 = 1.9598795012568466$, respectively.

3.1. Case I: Uniform Type Input Point Source

Figures 1-5 demonstrate the concentration patterns for a uniform input point source of heterogeneous porous media described by analytical solution (23, 24, 25).

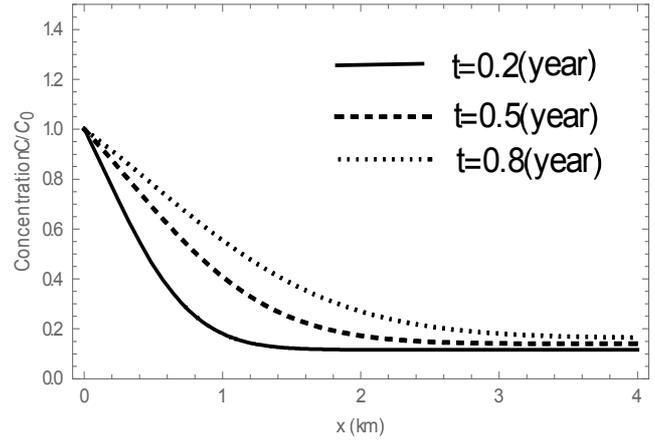


Figure 1. The contaminant concentration distribution profile obtained in (23) for various time $t(\text{year}) = 0.2, 0.5, 0.8$ in the time domain $0 < t \leq t^*$.

Figure 1. illustrates the dimensionless concentration distribution for different time $t(\text{year}) = 0.2, 0.5, 0.8$ and for fix retardation factor $R = 1.15$ and dispersion coefficient $D_0(\text{km}^2 \text{year}^{-1}) = 0.67$. It reveals the concentration distribution pattern produced by a uniform continuous point source in time domain $0 < t \leq t^*$. Concentration levels near to the source boundary are observed to be higher for higher time, lower for lower time, and stabilize after a distance away from the origin.

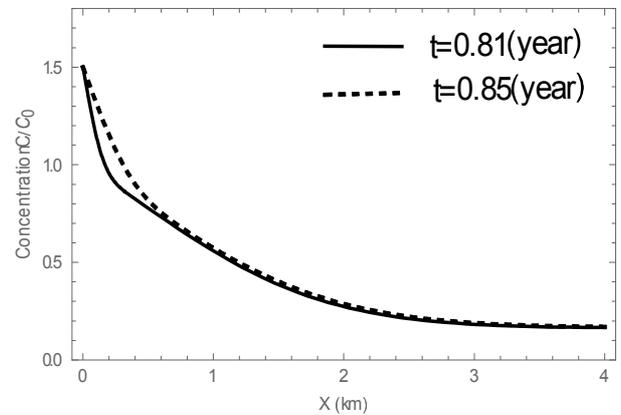


Figure 2. Contaminant concentration profiles obtained in (24) for various time $t(\text{year}) = 0.81, 0.85$ in the time domain $t^* < t \leq t_{\xi}^*$.

Figure 2. depicts the concentration distribution at different time $t(\text{year}) = 0.81, 0.85$ in the time domain $t^* < t \leq t_{\xi}^*$, assuming retardation factor and dispersion coefficients $R = 1.15, D_0 = (\text{km}^2 \text{year}^{-1}) = 0.67$ respectively and other parameters are kept fixed. Groundwater velocity and dispersion coefficient vary rapidly with time, depending on unsteady parameters, as in the previous time domain. A sudden increase of amount 0.5 can be seen at both values of time at $x = 0$ and then it drops rapidly and reaches a steady

state as we move away from the source. Due to some natural phenomena described in the previous section, the sudden release of the solute concentration in this time domain causes the input source concentration level to spike rapidly at the source. On comparing pattern of both the value of time, it can be observed that sudden release concentration traversing into the medium with time and move in the direction of flow.

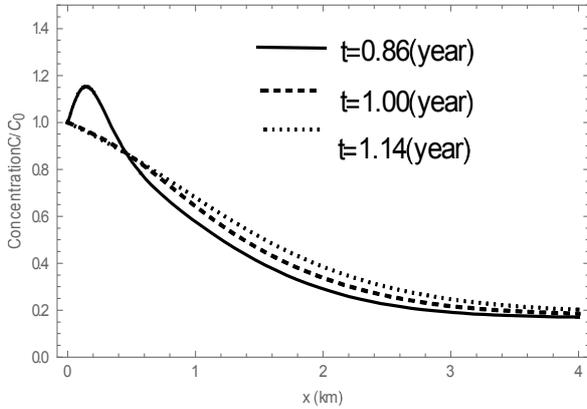


Figure 3. Contaminant concentration profile obtained in (25) for various time $t(\text{year}) = 0.86, 1.00, 1.14$ in the time domain $t_{\xi}^* < t$.

Figure 3. exhibits the pattern of solute concentration distribution at different time $t(\text{year}) = 0.86, 1.00, 1.14$ in the time domain $t_{\xi} < t$, assuming dispersion coefficient and retardation factor $D_0(\text{km}^2 \text{year}^{-1}) = 0.67$ and $R = 1.15$, respectively and other significant parameters are maintained constant. A pulse of concentration can be observed at time $t(\text{year}) = 0.86$, nearly at position $x(\text{km}) = 0.2$, then drops sharply with position and reaches a steady state. It is because, a sudden drop in the source concentration. The pulse dissolves with the time like $t(\text{year}) = 1.00, 1.14$ due to dispersion and convection phenomena. It can be observed that away from the source concentration is higher for higher time.

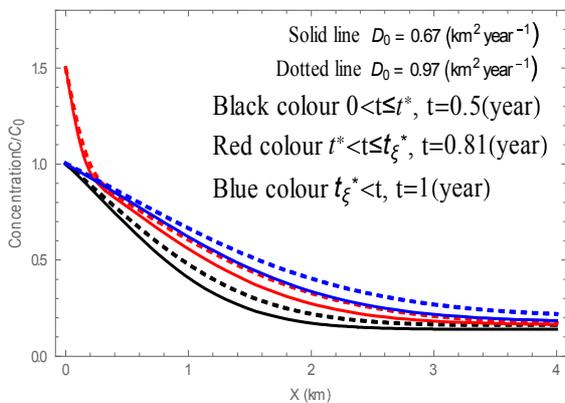


Figure 4. Concentration distribution profile for various dispersion coefficient $D_0(\text{km}^2 \text{year}^{-1}) = 0.67, 0.97$ in time domain $0 < t \leq t^*$, $t^* < t \leq t_{\xi}^*$, $t_{\xi}^* < t$, obtained with (23, 24, 25), respectively.

Figure 4. shows the effect of variation of different dispersion coefficient $D_0(\text{km}^2 \text{year}^{-1}) = 0.67, 0.97$ on concentration profile in different time domains $0 < t \leq t^*$, $t^* < t \leq t_{\xi}^*$, $t_{\xi}^* < t$ keeping the other parameters constant. It can be observed that the concentration patterns in each domain are higher for higher dispersion coefficient. This is due to the fact that dispersion phenomenon causes the spreading of the plume in the medium.

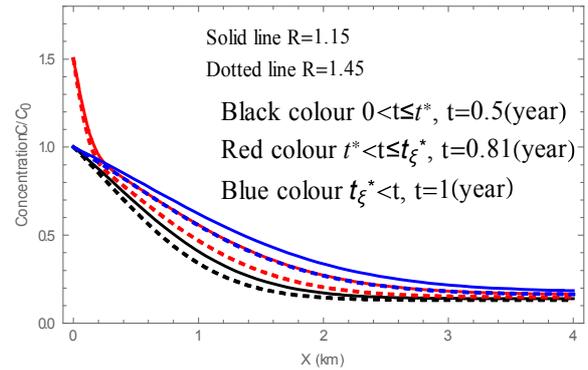


Figure 5. Concentration distribution profile for various retardation factors $R = 1.15, 1.45$ in time domain $0 < t \leq t^*$, $t^* < t \leq t_{\xi}^*$, $t_{\xi}^* < t$ obtained with (23, 24, 25), respectively.

Figure 5. shows the effect of various retardation factors $R = 1.15, 1.45$ on the concentration distribution in different time domains $0 < t \leq t^*$, $t^* < t \leq t_{\xi}^*$, $t_{\xi}^* < t$ for dispersion coefficient at $D_0(\text{km}^2 \text{year}^{-1}) = 0.67$, keeping all other parameters fixed. It can be seen that at a particular position in each time domain, the concentration level decreases with an increment in retardation factor.

3.2. Case II: Varying Type Input Point Source

Figures 6, 7, 8, 9 and 10 display the solute concentration patterns for different input sources in the heterogeneous porous medium described by (30, 31, 32).

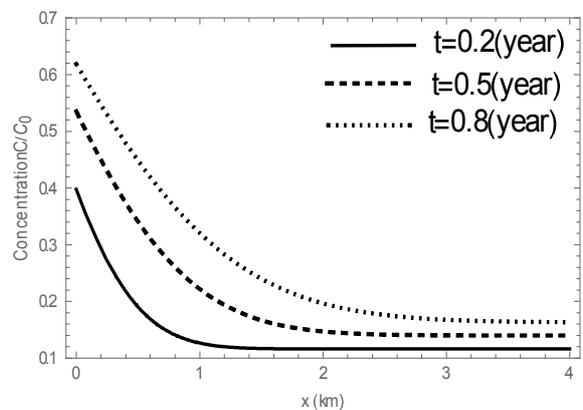


Figure 6. Contaminant concentration profile obtained in (30) for various time $t(\text{year}) = 0.2, 0.5, 0.8$ in the time domain $0 < t \leq t^*$.

Figure 6 describes the dimensionless concentration distribution at different time $t(\text{year}) = 0.2, 0.5, 0.8$ in the time domain $0 < t \leq t^*$, assuming other parameters fixed. Unlike a uniform input source, it shows the concentration value at the origin (at the source), which is lower at lower time and higher at higher time. Contaminant concentrations continued to decline with position, but across the domain, their trends remain higher for higher time and lower for lower time.

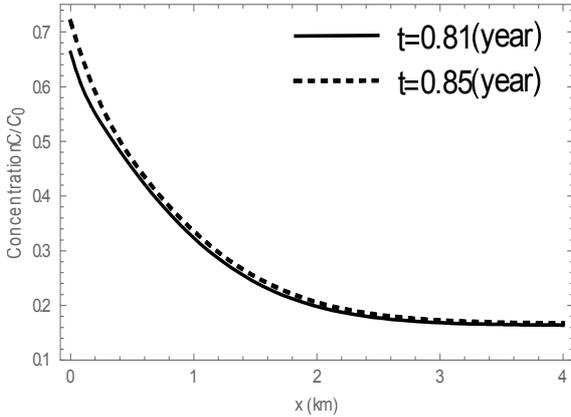


Figure 7. Contaminant concentration profile obtained in (31) for different times $t(\text{year}) = 0.81, 0.85$ in the time domain $t^* < t \leq t_\xi^*$.

Figure 7. shows the concentration profile at different times $t(\text{year}) = 0.81, 0.85$ in the time domain $t^* < t \leq t_\xi^*$ for retardation $R = 1.15$, $D_0(\text{km}^2 \text{year}^{-1}) = 0.67$, keeping other parameters fixed. The concentration level is steadily decreasing with the position and stabilizing toward to the opposite end, but near the source of the domain, its depreciation is higher for higher time and lower for lower time. On comparing the uniform input source shown in Figure 2., in this time domain ($t^* < t \leq t_\xi^*$) there is a moderate increment in the source concentration in present case in comparison to uniform input source case (Case 1). Near the source boundary, the concentration level remains high for a higher time and reaches a steady state beyond $x = 3(\text{km})$.

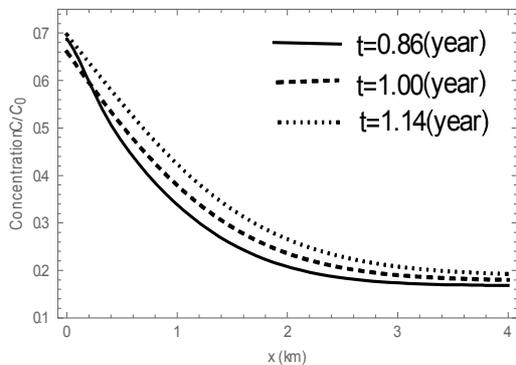


Figure 8. Contaminant concentration profile obtained in (32) for various time $t(\text{year}) = 0.86, 1.00, 1.14$ in the time domain $t_\xi^* < t$.

Figure 8. shows the concentration profile in the time domain $t_\xi^* < t$ at different time $t(\text{year}) = 0.86, 1.00, 1.14$. It can be observed that the solute concentration patterns near the boundary are different at different times but the trends are almost the same after a distance from the origin which is inherently decreasing relative to position. Significant changes can be observed in the concentration pattern over time. The concentration levels at a particular position remain higher at higher time and lower at lower time.

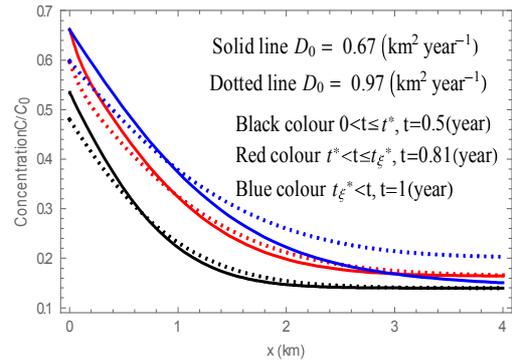


Figure 9. Contaminant concentration profile obtained in (30, 31, 32) for different dispersion values $D_0(\text{km}^2 \text{year}^{-1}) = 0.67, 0.97$ in time domain $0 < t \leq t^*$, $t^* < t \leq t_\xi^*$, $t_\xi^* < t$.

Figure 9. shows the effect of different dispersion coefficient values $D_0(\text{km}^2 \text{year}^{-1}) = 0.67, 0.97$ on the concentration distribution in each time domain $0 < t \leq t^*$, $t^* < t \leq t_\xi^*$, $t_\xi^* < t$, assuming a retardation factor $R = 1.15$ and other parameter values fixed. It can be seen that for a higher dispersion coefficient in each time domain, the value of the concentration pattern remains lower near the inlet source, but order reverses on moving away from the source. This is due to the fact that the increased dispersion causes extra solute mixing in the domain. The concentration level attenuates at the other end by almost the same level.

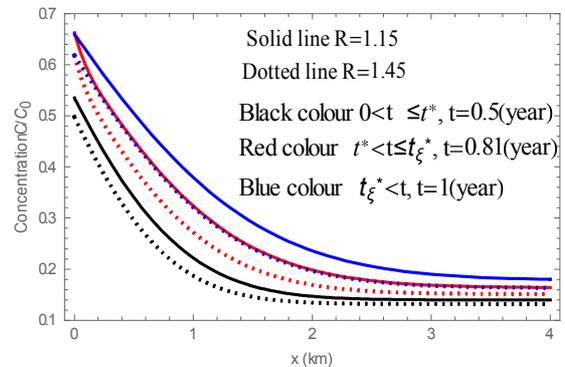


Figure 10. Contaminant concentration profile obtained in (30, 31, 32) for different retardation factors $R = 1.15, 1.45$ in time domain $0 < t \leq t^*$, $t^* < t \leq t_\xi^*$, $t_\xi^* < t$.

Figure 10 depicts the concentration distribution profile for different retardation factor $R = 1.15, 1.45$ in different time domains ($0 < t \leq t^*$, $t^* < t \leq t_{\xi}^*$, $t_{\xi}^* < t$) assuming dispersion coefficient $D_0 (km^2 year^{-1}) = 0.67$ and keeping other effective fixed. It can be observed that the concentration level within the domains decreases with an increment in retardation factor in each time domain. It is observed that in all time domains close to the outflow boundary, the variation in the concentration distribution profile is quite significant the near inlet boundary.

4. Conclusion

Analytical solutions of the advection-dispersion equation (ADE) are obtained under the assumptions of scale- and time-dependent dispersion to assess groundwater contamination in heterogeneous semi-infinite porous structures. Solutions were developed for two types of input sources (uniform and varying nature). The model also takes into account the effects of heterogeneity parameters, first-order decay and zero-order production. The proposed solution of ADE can effectively assess soil and groundwater contamination affected by the position and time-dependent parameters as well as space-dependent source boundary conditions. One can find the concentration of the pollutant at any time and place in the aquifer domain by using the solution obtained. The concentration level is high near the source boundary, but as the distance to the source increases, the concentration level trend continues to fall and attains the steady state. With important characteristics like dispersion and groundwater velocity, as well as medium heterogeneity, the influence of time seems to predominate. The solution is obtained with the help of the Laplace Integral Transform Technique. This problem offers a more realistic approach to variable dispersion and groundwater velocity because these values may be supposed to be different in different time intervals. These results highlight the importance of the dispersion function in the transport of solutes in a heterogeneous porous medium. The obtained results show that the pollutant transport is mainly influenced by the temporal input source. The developed mathematical model can be considered an effective tool to understand the transport behavior of pollutants in the surface and groundwater phenomena.

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