**Pure and Applied Mathematics Journal**
2022; 11(4): 51-69
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20221104.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

# Proof of the Beal, Catalan and Fermat's Last Theorem Based on Arithmetic Progression

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**To cite this article:**
Mohammad Amin Sharifi. Proof of the Beal, Catalan and Fermat's Last Theorem Based on Arithmetic Progression. *Pure and Applied Mathematics Journal*. Vol. 11, No. 4, 2022, pp. 51-69. doi: 10.11648/j.pamj.20221104.12

**Received**: June 25, 2022; **Accepted**: July 18, 2022; **Published**: July 26, 2022

**Abstract:** The Beal's Conjecture, Catalan's Theorem and Fermat's Last Theorem are cases within the number theory's field, so the Beal's Conjecture states that: $A^X+B^Y=C^Z$ where $A$, $B$, $C$ are positive integers and $X$, $Y$ and $Z$ are all positive integers greater 2, then $A$, $B$ and $C$ have a common prime factor and Catalan's Theorem express that $Y^p=X^q+1$ when $X$, $Y$, $q$, $p$ are integer numbers and greater than one, this equation has not solution in integer numbers exception to 8, 9 numbers. The Fermat's Last Theorem has the form of the Beal's Conjecture when $X$, $Y$, and $Z$ equal to $n$, then states that impossible find any solution in integer numbers for this equation. This article presents the proof of the Beal, Catalan and Fermat's Last Theorem, and generalizes these theorems have relationship with arithmetic sequence that this sequence outcome from subtraction of exponent integer numbers between successive terms. Then illustrated an exponent integer numbers built from two parts: one of the progression and other the non-progression, when a Diophantine equation has square power we dealt with summation of one series of arithmetic sequence that can increase terms of a progression by other progression. Thus can find relationship between Pythagoras' equation, Catalan and Fermat-Catalan's equation that obtained from Pythagoras' equation ($a^2+b^2=c^2$), the other word Catalan and Fermat-Catalan's equation a form of Pythagoras' equation when displace a point on the circle, at that time Pythagoras' equation reform to Catalan and Fermat-Catalan's equation. And also the Beal's Conjecture when $A$, $B$, $C$ are coprime, another form of Fermat's Last Theorem that both dealt with summation of several series of arithmetic progression, that impossible increase terms of a progression by other progression or a several series of sequence that shape is similar to a triangular that represented rows of progression and with a non-sequence parts that must change to sequences which rows of this progression less than initial progression. Also determine the Fermat's Last Theorem has no solution in integer number then Beal's Conjecture when $A$, $B$, $C$ are coprime also has no solution in integer number. The last term provides rules of Beal's Conjecture for solution and determine that this conjecture is super circles that obtained from primary circles, this primary circles existed from Catalan's Theorem, Fermat-Catalan's theorem and other forms. All of primary circles are based on the Pythagoras' equation and right triangle.

**Keywords:** Beal's Conjecture, Catalan's Theorem, Fermat's Last Theorem, Pythagoras' Equation, Arithmetic Progression, Prime Numbers, Integer Numbers

## 1. Introduction

The Beal's Conjecture was formulated in 1993 by Andrew Beal, a banker and amateur mathematician, while investigating generalization of Fermat's last theorem. After 4 years in 1997 he publicly announced to offer a $5,000 prize money for the peer review proof. By now he increased the prize money to sum of $1,000,000. The Beal's Conjecture is a proposition within the number theory's field, the conjecture states that: $A^X+B^Y=C^Z$ where $A$, $B$, $C$ are positive integers and $X$, $Y$ and $Z$ are all positive integers greater 2, then $A$, $B$ and $C$ have a common prime factor [10, 2-6]. Stated another way, there is no solution in integer for $A^X+B^Y=C^Z$ in the case of $A$, $B$, $C$, $X$, $Y$, $Z$ positive integer and $X$, $Y$, $Z > 2$ if $A$, $B$ and $C$ are coprime [1].

As in Porras Ferreira, Andy Beal studied equations with independent exponents. He worked on several algorithms to generate solution sets, but the nature of the algorithms he developed required a common factor in the bases. He suspected that using coprime bases might be impossible and set out to test his hypothesis. With the help of computers and a colleague. He tested this for all variable values up to 99. Many solutions were found, and all had a common factor in the bases. Andy Beal wrote many letters to mathematics periodicals and number theorists. Dr. Harold Edwards from the department of mathematics at New York University and author of "Fermat's Last Theorem, a genetic introduction to algebraic number theory, confirmed that the discovery was unknown and called it "quite remarkable". Dr. Earl Taft from the department of mathematics at Rutgers University relayed Andy Beal's discovery to Jarell Tunnell who was "an expert on Fermat's Last Theorem", according to Taft's response, and also confirmed that the discovery and conjecture were unknown [7].

According to Stewart and Tall, the Ancient Greeks, though concentrating on geometry, continued to take an interest in numbers. In c. 250 A. D. Diophantus of Alexandria wrote a significant treatise on polynomial equations which studied solutions in fractions. Particular cases of these equations with natural numbers solutions have been called Diophantine equations to this day [8]. The Pythagorean theorem is also including in Diophantine equations has been used around the world since ancient times, it is named after the Greek philosopher and mathematician, Pythagoras, who lived around 500 BC and show relationship among the lengths of the three sides of a right triangle. This states that in any right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse [9]. The other Diophantine equation is Fermat's Last Theorem, so Pierre de Fermat wrote a note in the margin of his copy of Diophantus' Arithmetical stating about in 17th century that is now popularly known as Fermat's Last Theorem (FLT) that $A^n+B^n=C^n$ has no solution in positive integer numbers when $n>2$ [12]. As in Mazaris, Fermat's Last Theorem is perhaps the single most famous mathematical problem of all times. Despite the extensive literature devoted to its proof and it remained unsolved for nearly 4 centuries [11].

Finally, after 374 years the famous theorem of Fermat was demonstrated in 150 pages by A. wiles [13]. But before The Andrew Wiles for $n$=4 proved by Fermat himself and using by the beautiful method of Descente infinite and $n$=3 demonstrated by Euler, Peter Lejeune Dirichlet dealt with the fifth and fourteen powers, Gabriel Lamé and Kummer for all powers to the 100th except 37, 59 and 67, and Friedrich Gauss tried to correct Lamé's attempt to the 7th power, but gave up after not getting success. The author also points out that in 1980 other mathematicians proved the theorem for all powers to the 125,000th [1, 14]. The Catalan's Theorem in number theory in one of those mathematical problems that are easy to formulate but extremely hard to solve. The conjecture predicts that 8 and 9 are the only consecutive perfects powers, in other words, the Diophantine equation $Y^p=X^q+1$ has no solution in integer numbers when $p$, $q > 1$ and $X$, $Y > 0$ [15, 16]. Catalan, at that time a teacher that his conjecture published an extract from a letter from the Belgian mathematician Eugéne Charles Catalan (1814-1894) to the editor. The conjecture has been open for more than 150 years, finally has been proved in 2002 by Preda Mihailescu [15, 17].

This article proved the Beal's Conjecture, Catalan and Fermat's Last Theorem by arithmetic progression and use the basic mathematics, as well as these conjectures are include in Diophantine equation and must have relation between themselves that this relationship represents by sequence. The deduction of Catalan, Fermat and Beal Conjecture's proof are divided into four parts. The first part consists of the progression of square of integer numbers and presents the Pythagoras's theorem equivalent to progression and by progression of square of integer numbers that can easily find Pythagorean triple. Also progression of cube of integer numbers make several series of arithmetic progression. The second part presents the deduction of the Catalan's Conjecture and proved in three parts: the analysis of cube and square of progression ($1+X^3=Y^2$), analysis of ($1+X^2=Y^3$) and equations ($1+X^2=Y^p$), ($1+X^q=Y^2$) and ($1+X^q=Y^p$).

The third part is proof of the Fermat's Last Theorem that are contains two subheadings: the first part illustrates the synthesis of progression and (FLT) that express increase one or more of row terms of progression when $n$=3, the second subheading discuss about progression when n>3. The last part demonstrated of the Beal's Conjecture include in two subheadings that the firs subheading showed relationship between the real solution of Fermat's equation and Beal's Conjecture that (FLT) has direct relation with Beal's equation and Beal's Conjecture dependence to (FLT) when $A$, $B$, $C$ are coprime. The second subheading apparent that situation $A$, $B$, $C$ when have coprime factor that outcome from primary triple that this triple has not coprime factor consists of Fermat-Catalan's theorem in equation $A^X+B^Y=C^2$ when $1/X+1/Y+1/2<1$, $1/X+1/Y+1/2>1$ and equal to one [18], and other special cases that all attain from Pythagoras's theorem and where the bases triple called primary circle. Also presents some solutions of the Beal's Conjecture in tables which this conjecture makes super circles than primary circles.

## 2. The Progression of Square and Cube of Integer Numbers

### *2.1. The Progression of Square of Integer Numbers*

The Pythagoras' Theorem, Fermat's Last Theorem, Catalan's Conjecture, Fermat-Catalan's Theorem and Beal's Conjecture all of include in Diophantine equation. In this part of Diophantine equation, base and power are both variable to exception of Pythagorean's theorem that exponent is square and main concept is that compound of two situations, the first situation is a form of power function and second status is a form of exponential function. Investigation and consideration of both power-exponential function are not comfort and easy, so for this reason study dedicated both situations separately. So analysis power function and then revert to power-exponential function. The first status a power function has a form of $cx^p$ and $p$ greater than one and base is variable, then verify this form demonstrated that come into being from a sequence. Now consider natural numbers $\mathbb{N}$=1, 2, 3, …, $n$ that equal to the base of power function and $p$=2. Then write equation (1),

$$x^2=1,4,9,16,25,36,49,64,81,100,...y_1 \qquad (1)$$

The equation (1) that exist from square of natural numbers and execute subtraction in between consecutive numbers, so become,

$$\Delta x^2 = 3,5,7,9,11,13,15,17,19,...\Delta N \tag{2}$$

In equation (2) inspect over that and apprehend which difference between successive terms is a constant so when a sequence has a constant difference between successive terms it is called an arithmetic progression [19]. Where the first term is 3 and common difference is 2. Denoted common difference to $d$ and the first term to $a_1$, then appear equation (3).

$$\Delta x^2 = a_1, a_1+d, a_1+2d, a_1+3d, ...a_1+(n-1)d \tag{3}$$

For sum of $n$ terms of an arithmetic progression has bellow equation,

$$S_n = n\left(\frac{a_1+a_n}{2}\right) = \frac{n[2a_1+(n-1)d]}{2} \tag{4}$$

Where $a_1$ is the first term and $a_n$ is the last term, illustrated in figure 1 that progression showed similar to rise of step. The figure 1 determine that sum of rise of step and additionally to one then equal to function $y=x^2$, now by adding one on sum of $n$ term of arithmetic progression, then outcome a formula that obtain square of ($n$+1) natural numbers.

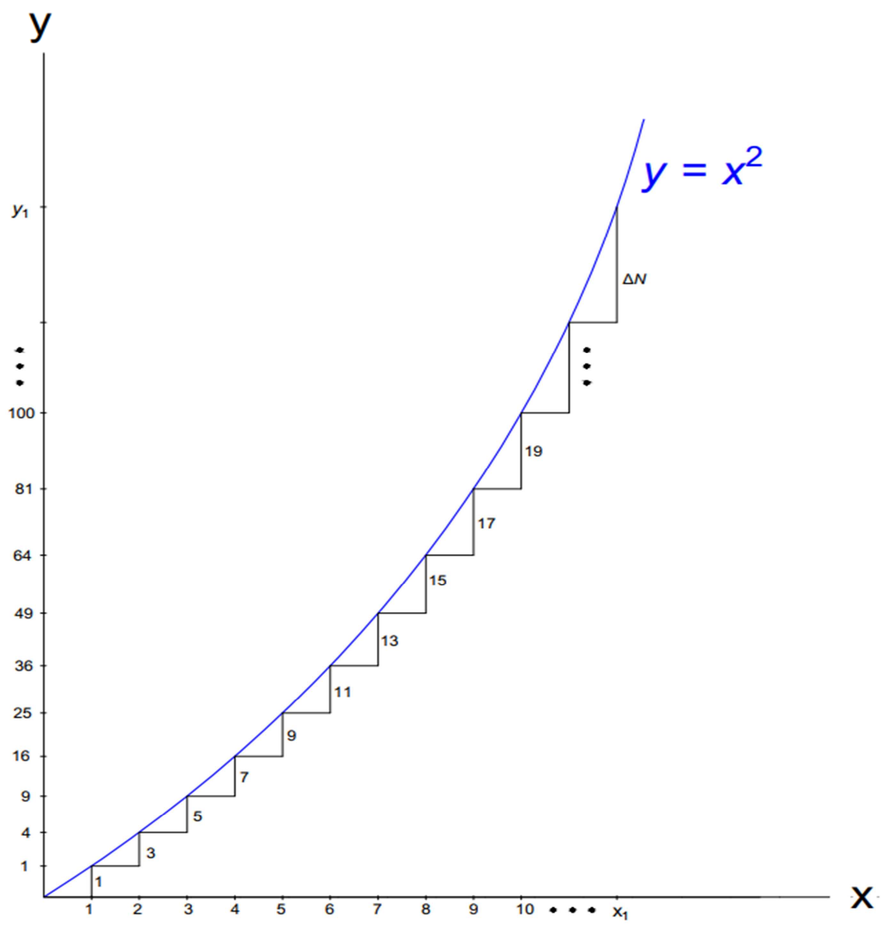


***Figure 1.*** *Illustrated the progression of square of integer numbers. Source: author.*

$$y = (n+1)^2 = \frac{n[2a_1+(n-1)d]+2}{2} \tag{5}$$

In equation (5) is obvious that combined of two section, one of that is series of sequence and anther of a constant, and this equation distinguish Pythagoras equation by equations (7) and (8) that Pythagoras's triple dealt with arithmetic progression.

$$a^2 + b^2 = c^2 \tag{6}$$

$$\frac{n[2a_1+(n-1)d]+2}{2} + \frac{m[2a_1+(m-1)d]+2}{2} = \frac{s[2a_1+(s-1)d]+2}{2} \tag{7}$$

$$(n+1)^2 + (m+1)^2 = (s+1)^2 \tag{8}$$

### *2.2. The Pythagoras Theorem Equivalent to Arithmetic Progression*

Actually equation (7) specify that $m$-th terms of arithmetic progression can increase $n$-th terms of progression that attain $s$-th terms of sequence. So for this equation we must remove constant part of $\frac{m\left[2a_1+(m-1)d\right]+2}{2}$ and convert to a series of progression that the first term of this series must greater than last term of $\frac{n\left[2a_1+(n-1)d\right]+2}{2}$ amount of the common difference, in fact converted equation $m^2+2m+1$ to form of $r^2+(a_1-1)r$. Also by equation (8) can find Pythagoras triples with slight transformation and with put two value in equation (9), resulting in:

$$n=\sqrt{(s+1)^2-(m+1)^2}-1 \tag{9}$$

And by simplification binomial of $(s+1)^2$, equation (9) become below equation.

$$n=\sqrt{s^2+2s+1-(m+1)^2}-1 \tag{10}$$

In equation (10) $2s+1$ must equal to $(m+1)^2$, which this equation infinite numbers can find which this equation satisfies. But difference between $n$ and $s$ are one value, if want to difference between $n$ and $s$ more than one. From $s^2$ reduce a specific amount of number and denoted specific number to $p$ then rewrite equation (10).

$$(s-p)^2=s^2-2sp+p^2 \tag{11}$$

And with compare of equation (10) and (11) outcome below equation.

$$n=\sqrt{(s-p)^2+2s(p+1)-(p^2-1)-(m+1)^2}-1 \tag{12}$$

In equation (12) infinite numbers are exist that $m=\sqrt{2s(p+1)-(p-1)}-1$ satisfy.

### *2.3. The Progression of Cube of Integer Numbers*

The power function with degree of two simply satisfy but important that solve power function $cx^p$ with $p > 2$, Now consider natural numbers with $p$=3

$$x^3=1,8,27,64,125,216,343,512,729,1000,\ldots y_1 \tag{13}$$

And subtracts in between consecutive numbers in equation (13), as follow:

$$\Delta x_1^{\ 3}=7,19,37,61,91,127,169,217,271,\ldots\Delta N_1 \tag{14}$$

Equation (14) clarify that no create a progression by a step of subtraction of consecutive numbers, that can be graphically represented in figure 2 and again execute subtraction of between successive terms in equation (14).

$$\Delta x_2^{\ 3}=12,18,24,30,36,42,48,54,\ldots\Delta N_2 \tag{15}$$

Equation (15) distinguish that make an arithmetic progression and where common difference is 6 and 12 is the first term of this progression. Thus can represent equation (14) and (15) graphically in figure 2 and 3.

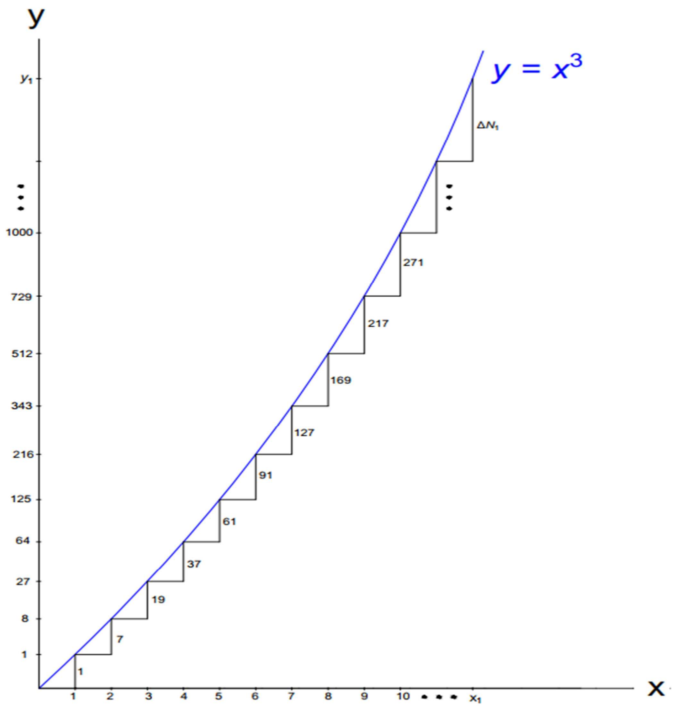


***Figure 2.*** *Graphic representation of subtraction cube of integer numbers between successive terms. Source: author.*

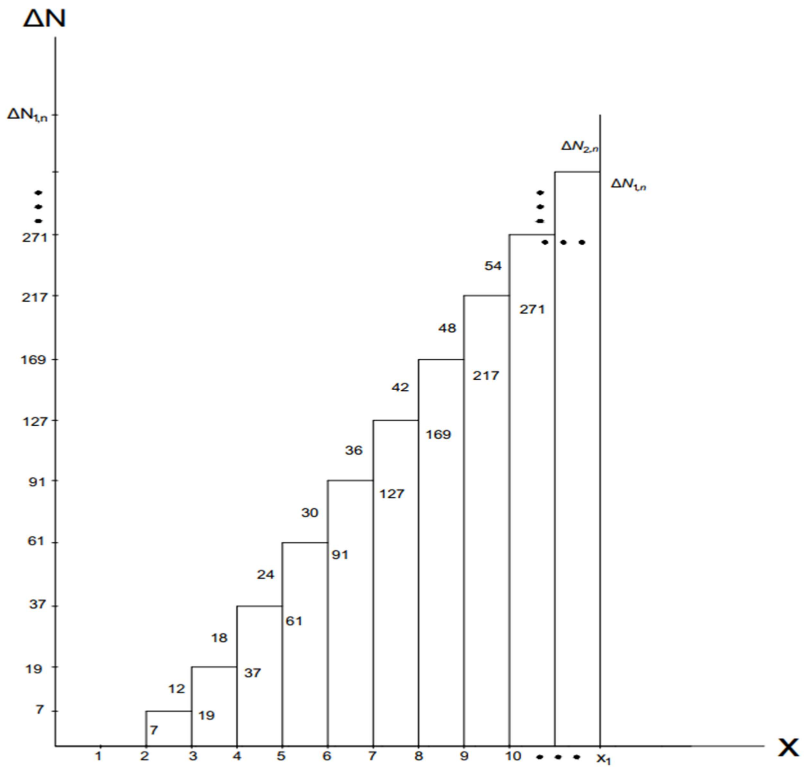


***Figure 3.*** *Graphic representation of progression that outcome from subtraction of cube of integer numbers between successive terms. Source: author.*

Pattern of histogram in figure 2 represented that obtain from equation (14) and figure 3 that outcome from subtraction of cube of consecutive between natural numbers that stipulate that exist from two parts, one of height of rectangular or height of non-progression and another difference height of histogram or part of progression. The destination is that find value of function $y=x^3$, so consider none progression part of histogram, if find the $n$-th height of rectangular must sum $n$ part of sequence and over add 7 on equation (16), as according to figure 3 that summation of progression less amount of seven.

$$\Delta N_{1,n} = \frac{n\left[2a_1+(n-1)d\right]}{2}+7 \tag{16}$$

Equation (16) demonstrate $n$-th height of rectangular but if invocated to find ($n$-1)-th height of rectangular must subtraction with before height of rectangular, that resulting of subtraction become $a_1+(n-1)d$, this equation is a term of arithmetic progression that can write equation (17).

$$\Delta N_{2,n} = \frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2} = a_1+(n-1)d \tag{17}$$

Amount of difference between $n$-th and ($n$-1)-th terms of progression equal to equation (17), then write equation (18) and (19).

$$\Delta N_{1,n-1} = \frac{n\left[2a_1+(n-1)d\right]}{2} - \Delta N_{2,n} \tag{18}$$

$$\Delta N_{1,n-1} = \frac{n\left[2a_1+(n-1)d\right]}{2} - \left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\}+7 \tag{19}$$

For another rectangular that exist of subtraction of cube of natural numbers and less then ($n$-1)-th terms of height of rectangular and sum of 1 up to ($n$-1)-th term of natural numbers of $\Delta N_{2,n}$ and every terms of natural numbers of $\Delta N_{2,n}$ that subtract from $\frac{n\left[2a_1+(n-1)d\right]}{2}$. Also sum of triangular progression numbers from 1 up to ($n$-2)-th terms of common difference from this progression. Thus investigate equation (16), (17) and (19) when find height of rectangular that less from $\Delta N_{1,n}$ and this procedure at firs find difference between $\Delta N_{1,n}$ and $\Delta N_{1,n-1}$, and also this procedure continue that so long as height of rectangular become generate $\Delta N_{1,1}$, where create a descending arithmetic progression that less amount of 6 numbers in every terms of sequence. The terms of $\Delta N_{2,n}$ less amount of one unit that clarify equation (21) and sum of ($n$-1)-th natural numbers. We can use for sum of natural numbers from equation (20). Resulting in:

$$S_n = \frac{n(n+1)}{2} \tag{20}$$

$$S\Delta N_{2,n} = \frac{n(n-1)}{2}\Delta N_{2,n} \tag{21}$$

$$\frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\} = S\Delta N_{2,n} \tag{22}$$

Consider equation (21) that is revers of figure 3 or ascending arithmetic progression that together make a larger of rectangular, but this equation determine that increase $a_1+(n-1)d$ amount of resulting product of natural numbers, but sequence change amount of one common difference between two consecutive term. Where demonstrate increase a value of common difference in equation (22). The extra value of common difference that must subtract from equation (22) until make descending progression, thus extra common difference in this situation create a triangular numbers sequence, so the triangular numbers are those counting numbers that can be written as $T_n=1+2+3+4+6+7+8+\ldots+n$ [20]. Thus create triangular numbers from common difference. Which this triangular numbers started from ($n$-2)-th term of sequence, so for sum of n-th triangular progression use from equation (23) [20]. In which the sum of triangular progression denoted to *TPN*.

$$TPN = \frac{n(n+1)(n+2)}{6} \tag{23}$$

The triangular progression numbers of common difference start from two units after than $n$-th terms of arithmetic sequence and deduce from equation (22) and (23) can write equation (24). Then denoted this equation to $(S\,\Delta N_{2,n})TPN$.

$$(S\Delta N_{2,n})TPN = \frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\} - \frac{n(n-1)(n-2)d}{6} \tag{24}$$

For every part of descending arithmetic progression sum on a seven number and for ($n$-1)-th terms of descending arithmetic progression sum on 7($n$-1). According to figure 3 that every height of histogram that there make sequence less amount of seven units. Then rewrite equation (24) to (25) and denoted to $(S\,\Delta N_{2,n})TPN_c$ where $c$ is for constant and non-progression part that add over of equation (24).

$$(S\Delta N_{2,n})TPN_c = \frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\} - \frac{n(n-1)(n-2)d}{6} + 7(n-1) \tag{25}$$

By compare of equation (18), (19), (23) deduce that subtract every term of equation (24) from $\frac{n\left[2a_1+(n-1)d\right]}{2}$ that attained a descending arithmetic progression and sum of 7($n$-1) on this equation, then denoted by $\Delta\varepsilon$ that appear equation (26).

$$\Delta\varepsilon = (n-1)\left\{\frac{n\left[2a_1+(n-1)d\right]}{2}\right\} - (S\Delta N_{2,n})TPN + 7(n-1) \tag{26}$$

And another form of equation (24) can write this form,

$$\Delta\varepsilon = (n-1)\left\{\frac{n\left[2a_1+(n-1)d\right]}{2}\right\} - \frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\} + \frac{n(n-1)(n-2)d}{6} + 7(n-1) \tag{27}$$

By sum of equation (16) and (26) that exactly attained all height of histogram that approach to function $y=x^3$, then outcome equation (28) and denoted to $\Delta\varepsilon_y$,

$$\Delta\varepsilon_y = \Delta N_{1,n} + \Delta\varepsilon \Rightarrow \Delta\varepsilon_y = \frac{n^2\left[2a_1+(n-1)d\right]}{2} - (S\Delta N_{2,n})TPN + 7n \Rightarrow$$

$$\Delta\varepsilon_y = \frac{n^2\left[2a_1+(n-1)d\right]}{2} - \frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\} + \frac{n(n-1)(n-2)d}{6} + 7n \tag{28}$$

Now see figure 2 and distinguish that arithmetic progression beginning of third rectangular and function $y=x^3$ more than equation (28) amount of two heights of rectangular, that this one height of rectangular its value of equal to 1, and also according to equation (16) when find height of $n$-th rectangular with sum 7 on its progression which this procedure repeats in every rectangular, so in the end remain a seven number which sum on with equation (28). Resulting in,

$$y=(n+2)^3=\frac{n^2\left[2a_1+(n-1)d\right]}{2}-\frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2}-\frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\}+\frac{n(n-1)(n-2)d}{6}+7n+8 \quad (29)$$

The equation (29) determine and represents another form of cube of an integer numbers that this equation combine of two prat, one of sequence and another of constant. Now can by equation (5) and (29) that both are power function and analyze the power-exponential function by progression of cube of integer numbers. So a power-exponential function has form of $cx^y$, and this form base and exponent both are variable. Then transform arithmetic progression of power-exponential integer numbers to Fermat's Last Theorem, Catalan and Beal's Conjecture for proof of those.

## 3. Proof of the Catalan's Conjecture

### 3.1. Analysis of Cube and Square of Progressions ($1+X^3=Y^2$)

The form of Catalan's Conjecture is very easy but one of the famous classical problem in number theory. This conjecture express that equation $Y^p=X^q+1$ when $X$, Y, $p$ and $q$ greater than one and can't find solution for $Y$, $X$ in integer number which this equation satisfy exception of these numbers ($X$, $Y$, $p$, $q$)=(2, 3, 2, 3). Now consider equation (5) and (29) for proof of Catalan's Conjecture, and transform equation (5) and (29) to equation $Y^p=X^q+1$. At first suppose $Y^p$ equal to equation (5) and $X^q$ equal to equation (29), where $p$, $q$=2, 3 respectively.

$$Y^2=(m+1)^2=\frac{m\left[2b_1+(m-1)d_s\right]}{2}+1 \quad (30)$$

$$X^3=(n+2)^3=\frac{n^2\left[2a_1+(n-1)d\right]}{2}-\frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2}-\frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right\}+\frac{n(n-1)(n-2)d}{6}+7n+8 \quad (31)$$

Equation (31) with sum of one must equal to equation (30), as fallow:

$$\frac{n^2\left[2a_1+(n-1)d\right]}{2}-\frac{n(n-1)}{2}\left\{\frac{n\left[2a_1+(n-1)d\right]}{2}-\frac{(n-1)\left[2a_1+(n-2)d\right]}{2}+\frac{n(n-1)(n-2)d}{6}+7n+8=\frac{m\left[2b_1+(m-1)d_s\right]}{2}+2 \quad (35)$$

$$1+(n+2)^3=\frac{m\left[2b_1+(m-1)d_s\right]}{2}+1 \quad (32)$$

In which $b_1$, $d_s$ equal to 3 and 2 respectively, then put value of $b_1$, $d_s$ on above equation and have,

$$(n+2)^3=\frac{m\left[2b_1+(m-1)d_s\right]}{2}\Rightarrow\frac{m\left[6+(m-1)2\right]}{2}$$
$$\Rightarrow\frac{2m^2+4m}{2}\Rightarrow m^2+2m$$

$$n=\sqrt[3]{m^2+2m}-2 \quad (33)$$

In equation (33) $m^2+2m$ must a complete root with grade of cube and $m$ that take a value in integer numbers which this equation satisfies, it is possible rewrite equation (33) to below equation,

$$n=\sqrt[3]{m^3\left(\frac{1}{m}+\frac{2}{m^2}\right)}-2 \quad (34)$$

Equation (34) indicate which this equation only has complete root when $m$ equal to two and other positive integer numbers $\frac{1}{m}+\frac{2}{m^2}<1$ and this equation has not answer. So from equation (30), (31) deduces that only part of progressive of equation (30) equal to constant portion of equation (31) and demonstrate that impossible equation 29 that has arithmetic progression, triangular numbers of common difference and with constant that change to an arithmetic sequence that its common and first term of this progression difference with other progression.

### 3.2. Analysis of Square and Cube of Progressions ($1+X^2=Y^3$)

Now consider another situation of Catalan conjecture that $X$ square and sum of with one that equal to $Y$ cube.

$$n=\sqrt[3]{m^2+2m+2}-2 \quad (36)$$

Another form of equation (36) can write in this kind,

$$n=\sqrt[3]{m^3\left(\frac{1}{m}+\frac{2}{m^2}+\frac{2}{m^3}\right)}-2 \quad (37)$$

In equation (37) $\frac{1}{m}+\frac{2}{m^2}+\frac{2}{m^3}$ never equal to one, in order to this equation has complete root.

### 3.3. Analysis of Power That Greater Than 3 and Square of Progressions: ($1+X^2=Y^p$), ($1+X^q=Y^2$) or ($1+X^q=Y^p$)

In third situation discuss Catalan's Conjecture when with $p$, $q$ exponent that $p$ or $q$ > 3, it is concept compare this situation with progression of power-exponential function. In fact, one power of term in Catalan's Conjecture equal to 2 then repeat equation (34), (37) with $p$ or $q$ degree of root that has not any solution in integer numbers. When $p, q > 3$ in this situation can transform $Y^p$ with $Y^{(p-3)}Y^3$ and also $X^q$ to $X^{(q-3)}X^3$, where base and exponent both are variable, and can write equation (38), in which $n, m$ less two value than $Y$ and $X$ respectively.

$$Y^{(p-3)}\left\{\frac{n^2\left[2a_1+(n-1)d\right]}{2}-\frac{n(n-1)}{2}\left[\frac{n\left[2a_1+(n-1)d\right]}{2}-\frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right]+\frac{n(n-1)(n-2)d}{6}+7n+8\right\}=$$

$$X^{(q-3)}\left\{\frac{m^2\left[2a_1+(m-1)d\right]}{2}-\frac{m(m-1)}{2}\left[\frac{m\left[2a_1+(m-1)d\right]}{2}-\frac{(m-1)\left[2a_1+(m-2)d\right]}{2}\right]+\frac{m(m-1)(m-2)d}{6}+7m+8\right\}+1 \quad (38)$$

Equation (38) is distinct that no possible this equation satisfy, as in this equation $Y^{(p-3)}(7n)$ must greater then $X^{(q-3)}(7m)$ amount of seven, and $n$-th terms of arithmetic and triangular progression must less than $m$-th terms of arithmetic and triangular progressions amount of one common difference until that Catalan conjecture satisfy which this situation impossible. also in the proof of (FLT) represented that all power-exponential function makes a progression which in its composition exist one value that without one cannot sequence of power-exponential integer numbers equal to other progression of power-exponential function. thus all situation of Catalan's Conjecture generalizes that no solution equation $Y^p=X^q+1$ exception of 8 and 9 numbers.

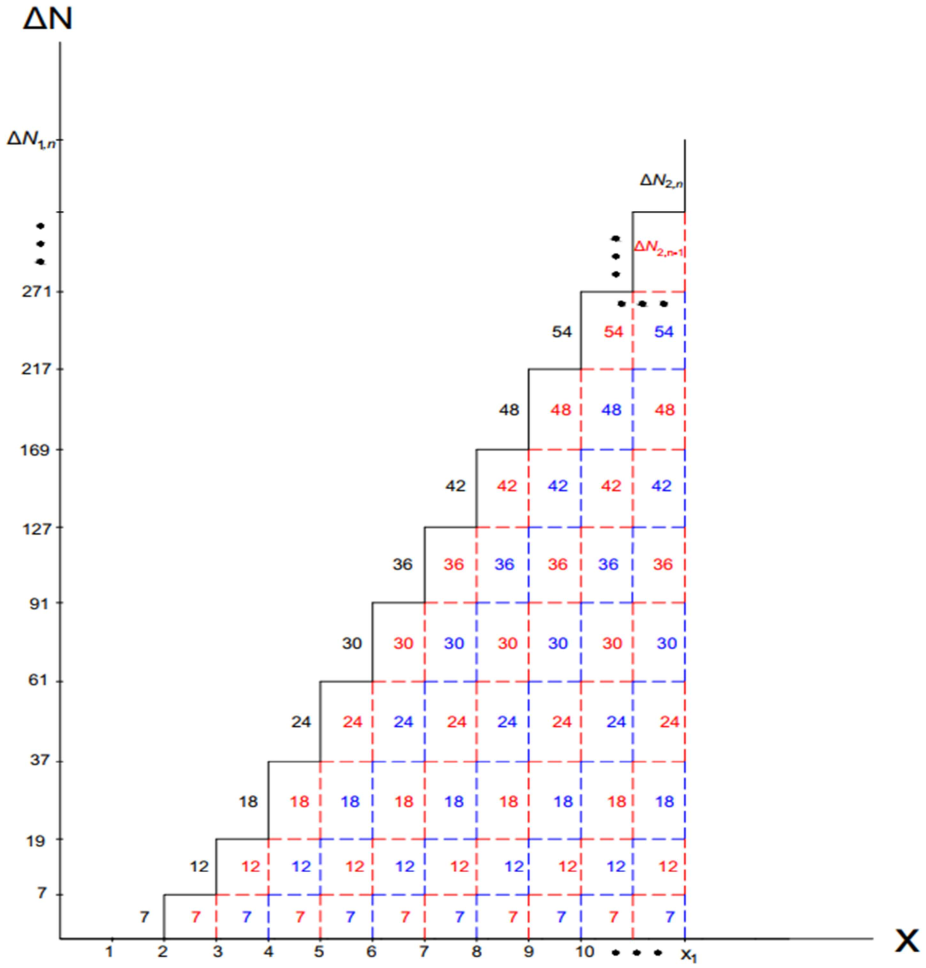


***Figure 4.*** *Illustrated the synthesis of progression that obtained from subtraction of cube of integer numbers between successive terms. Source: author.*

# 4. Proof of Fermat's Last Theorem

## *4.1. The Synthesis of Progression and Fermat's Last Theorem*

Fermat's Last Theorem is also a famous classical problem in number theory, this theorem declares that equation $A^n+B^n=C^n$ has no solution in integer numbers when $n>2$. So for proof of (FLT) use from sequence that find cube of positive integer numbers, at first determine type of synthesize of equation (29) by figure 4 that illustrated which the first progression with black color and other progression that located in under of the first progression showed with red and blue color. This figure determines that red and blue progression one term decreases than above successive row sequence, this reduce appear at end term of every progression. So can write equation (39) according with figure 4.

$$(n+2)^3=\left\{\begin{array}{ll} a_1+(a_1+d)+(a_1+2d)+\ldots+\left[a_1+(m)d\right] & \\ a_1+(a_1+d)+(a_1+2d)+\ldots+\left[a_1+(m)d\right] & \\ . & +\ldots+\left[a_1+(n-3)d\right]+\left[a_1+(n-2)d\right]+\left[a_1+(n-1)d\right] \\ a_1+(a_1+d)+(a_1+2d)+\ldots+ & +\ldots+\left[a_1+(n-3)d\right]+\left[a_1+(n-2)d\right] \\ a_1+(a_1+d)+(a_1+2d)+\ldots+ & . \\ . & \left[a_1+(m-2)d\right]+\left[a_1+(m-1)d\right]+\left[a_1+(m)d\right] \\ a_1+(a_1+d)+(a_1+2d)+(a_1+3d) & \left[a_1+(m-2)d\right]+\left[a_1+(m-1)d\right] \\ a_1+(a_1+d)+(a_1+2d) & \\ a_1+(a_1+d) & \\ a_1 \qquad\qquad + \qquad\qquad 7n+8 & \end{array}\right\} \tag{39}$$

Equation (39) demonstrated type of synthesis of cube of positive integer numbers and shape of this equation like is to triangular that the first row and column are equal with together according to amount of terms and all rows in summing together that where we betake from plus sign in between every row in equation (39) and other equations that come after this equation which similar to this equation. So transform (FLT) to power function $A^{(n-3)}A^3+B^{(n-3)}B^3=C^{(n-3)}C^3$ and can write $A^{(n-3)}$, $B^{(n-3)}$, $C^{(n-3)}$ to $R_1$, $R_2$, $R_3$ respectively, in fact power-exponential function changed to power function and can write below equation.

$$R_1A^3+R_2B^3=R_3C^3 \tag{40}$$

In equation (40) when $R_1$, $R_2$, $R_3=1$ this equation changes to the third degree equation and for proof of (FLT) use from smallest triple exponential integer number and equation (39). The smallest triple exponential integer numbers are Catalan numbers, then increase 1 to amount of smallest number of equation $A^3+B^3=C^3$, where assume $A^3$ is smallest number and then increase 8 to amount of $B^3$. In fact, $B^3$ equal to $A^3$ with product and addition of an integer number in which addition number is corrector that serialize sequence. This method for moving a small integer number to a greater integer number, and this procedure as follow.

$$\left[1+(A^3-1)\right]+\left[8+Cr+TA^3\right]=8+Cr+(T+1)A^3 \tag{41}$$

Where $T$ is amount of difference terms between $[1+(A^3-1)]$ and $[8+Cr+TA^3]$, when $T$ is an exponent number and power of $T$ was multiple of 3 that can represent by $t^{3p}$ in which multiple of 3 showed by $p$ and then $Cr$ equal to 8 with negative sign. In this status (FLT) has common factor, and then can transform equation (41) to equation (42).

$$A^3+\left(t^pA\right)^3=\left(t^{3p}+1\right)A^3 \tag{42}$$

Otherwise that $T$ is not difference terms of between $[1+(A^3-1)]$ and $[8+Cr+tA^3]$, in this situation $8+Cr$ complete imperfect terms of arithmetic progression and Fermat theorem has not common factor and $8+Cr$ represent by $Cr_1$, then outcome equation (43).

$$A^3+(Cr_1+TA^3)=Cr_1+(T+1)A^3 \tag{43}$$

#### 4.1.1. Increase One Row Terms of Progression

In equation (42) if $(t^pA)^3$ equal to $B^3$, then consider $(t^{3p}+1)A^3$ with equation (39) where $A^3$ must increase $B^3$ one terms of progression that outcome $C^3$ and satisfy equation (42). In this status must reform equation (39) and figure 4 that represent combination of arithmetic progression of cube of positive integer numbers, that demonstrate has a shape of triangular and change to a row of progression or combine like trapezoid. resulting in,

$$(n+2)^3=a_1+(a_1+d)+(a_1+2d)+\ldots+\left[a_1+(n-1)d\right]+\ldots+\left[a_1+(r-2)d\right]+\left[a_1+(r-1)d\right]+d+1 \tag{44}$$

In equation (44) $[a_1+(n-1)d]$, $[a_1+(r-2)d]$ represent respectively largest terms of progression $A^3$, $B^3$ in longest row, and formula for sum of equation (44) can write below equation.

$$(n+2)^3=\frac{r\left[2a_1+(r-1)d\right]}{2}+d+1 \tag{45}$$

Actually, equation (29), (39) must transform to equation (45) and (44) respectively and $7n$ in equation (29) and (39) can write form of common difference, so can write $7n$ to $nd+n$, thus equation (44) and (45) must greater one term than the longest row term of equation (29) and (39) that until increase $B^3$ amount of one row terms of arithmetic progression in every rows that outcome $C^3$. Now investigate equation (45), is possible which this equation outcome from equation (29)? At present transform equation (45) to value of $r$ and then have below equation,

$$(n+2)^3=3r^2+9r+7 \tag{46}$$

Equation (46) demonstrated $3r^2+9r+7$ is not equal to a cube of integer numbers. so for every positive integer numbers are impossible. Now $A^3$ make into two or more row terms of sequence as far as create an imperfect triangular and then investigate that have any solution in positive integer numbers or no. so inquire $nd + n$ in equation (29) that is a form of $7n$, when $n$ that take a value in integer numbers that like able divide to $d$ that represents by this formulaic $\frac{(n-5)d}{d} = \overset{+}{\mathbb{Z}} d$, if subtract 5 from $n$ and then 5+8 make $2d+1$, where $d+1$ need to increase for non-progression terms for equation (45). So equation (45) determine that less amount of common difference that complete a cube of integer numbers and denoted decrement of common difference by $\varepsilon d$, so can write equation (47) that has solution in positive integer numbers.

$$(n+2)^3 = 3r^2 + 9r + 7 + \varepsilon d \tag{47}$$

#### *4.1.2. Increase Rows Terms of Progression*

The equation (47) confirm a row terms of progression, now divide $A^3$ to more than one row terms. So operate this procedure that until $\varepsilon d$ become zero in the imperfect triangular form of equation (39) and this is my hypotheses that can zero $\varepsilon d$ before that triangular completed, but in the opposite appear an extra value that smaller than common difference. That because it is which for attain $nd+n$ use from extra common difference, in this situation extra value that from $d$ exist for increase of $n$ and until that extra value become zero has no solution like periodic and denoted extra value to $\varepsilon x$ and become equation (48),

$$(n+2)^3 = \left\{\begin{array}{l} a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(s-1)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(s-2)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(s-3)d] \\ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r+2)d]+[a_1+(r+3)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r+1)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r)d] \ + \ hd+h+\varepsilon x \end{array}\right\} \tag{48}$$

In equation (48) largest term of $A^3$, $B^3$ demonstrated by $[a_1+(n-1)d]$, $[a_1+(r-1)d]$ respectively, the $[a_1+(s-1)d]$ is the largest term of longest row and $[a_1+(r)d]$ is the largest term of shortest row of progression that outcome from impair of triangular that exist of $A^3$ and $h$ determine the amount of rows progression terms. So equation (48) has no solution so long as be existing $\varepsilon x$ but can zero $\varepsilon x$ like to periodic (7, 13, 19, 25, 31, …, $7+6p$). now investigate that can attain an equation $\varepsilon x$, $\varepsilon d$ both be zero or no, similar to below equation,

$$(n+2)^3 = \left\{\begin{array}{l} a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(z-1)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(z-2)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-1)d] \ +...+[a_1+(r-1)d]+...+[a_1+(z-3)d] \\ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r+2)d]+[a_1+(r+3)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r+1)d] \\ a_1+(a_1+d)+(a_1+2d)+...+[a_1+(n-2)d] \ +...+[a_1+(r)d]+ \ (7+6p)d+(7+6p) \end{array}\right\} \tag{49}$$

The synthesize of sequence in equation (49) site to similar a shape of rectangular and triangular. For deduce of above equation which this is possible or impossible, now consider the end of all terms of rows progression that site like an incline line. So the extra terms of progression than initial amount of terms in every rows that site on incline line must equal to disappear terms of progression of power function that compound like triangular and use for increment of terms in equation (49) and the largest term of this compound triangular of arithmetic progression be equal to $[a_1+(m-1)d]$. If assume counts of difference terms between $[a_1+(r-1)d]$, $[a_1+(z-1)d]$ equal to $k$ terms and the first term be $[a_1+rd]$ that equal to $b_1$ then have equation (50).

$$(m+2)^3 = \frac{(7+6p)_{,1}\left[2b_1+\left((7+6p)_{,1}-1\right)d\right]}{2} + \frac{(7+6p)_{,2}\left[2(b_1+d)+\left((7+6p)_{,2}-1\right)d\right]}{2} + \frac{(7+6p)_{,3}\left[2(b_1+2d)+\left((7+6p)_{,3}-1\right)d\right]}{2} + ... + \frac{(7+6p)_{,k}\left[2\left(b_1+(k-1)d\right)+\left((7+6p)_{,k}-1\right)d\right]}{2} \tag{50}$$

In which $(7+6p)_{,k}$ represents the $k$-th terms of progression that sum with together and difference between smallest terms of incline row with successive row is 6 or that common difference which use in progression of cube of positive integer numbers, and other form of equation (50) can write to form of equation (51).

$$(m+2)^3 = \frac{(7+6p)}{2}\left[k(k+1)b_1 + k(k-1)d + \frac{k(k+1)(7+6p)d}{2}\right] \tag{51}$$

In equation (51) $b_1$ has able divide to common difference and denoted by $\lambda d$ and this equation factorization by three parenthesize and so can write below equation,

$$(m+2)^3 = (7+6p)(d)\left[\frac{k(k+1)\lambda}{2} + \frac{k(k-1)}{2} + \frac{k(k+1)(7+6p)}{4}\right] \tag{52}$$

The deduce from equation (52) impossible that can write a cube of positive integer numbers to form of this equation, so apprehend that equation (43) and (42) have not any solution in integer numbers.

### *4.2. Increase of Row Terms of Progression When n > 3*

Now consider equation (40) with (42), (43) when $R_1$, $R_2$, $R_3 \neq 1$ and $R_1$, $R_2$, $R_3$ be an exponential integer numbers, then have a power function with exponent greater than 3. So arrangement and formation of an arithmetic sequence is feature of exponential integer numbers that appear when subtract between successive terms of exponential integer numbers and continue this procedure until that outcome a progression. The common difference of progression of $n$-th exponential of an integer numbers equal to factorial of $n$-th natural numbers that can represent by $(n)!$ and also $(n-1)$-th times subtract successive terms together that obtain a progression and the first term of this progression for $n$ (power) even numbers are equal to $2a_1/d=(n+1)$ and for $n$ odd numbers first term of progression equal to $a_1/d=(n-en)$ in which $en$ represented the counts of even numbers that appear in between 1 to $n$, for example $n$ for 7 three even numbers are being exist between 1 to 7 that $en$=3 and then $a_1/d$=4. The diagram of an exponential integer numbers until that outcome a progression is like tree, that have stem, branches and leaves. In which leafs is that progression and can write $n$-th exponential integer numbers by below equations.

$$A^n = 1 + \Delta A_{1,1} + \Delta A_{1,2} + \Delta A_{1,3} + \Delta A_{1,4} + \ldots + \Delta A_{1,A-1} \tag{53}$$

In which $\Delta A_{1,A-1}$ represents the term of difference of $(A-1)$-th subtraction of exponential integer number that less one term than $A$-th exponential integer numbers. The sum of second subtract successive terms of equation (53) addition of the first term of equation (53) equal to $\Delta A_{1,A-1}$. In fact, the sum of second, third and $(n-1)$-th subtraction of successive terms less amount of the first term of its above row than the last term of its above row and can transform and formulaicably equation (53) to form of triangular that all rows are summing. Resulting in,

$$A^n = \begin{bmatrix} \Delta A_{2,1} + \Delta A_{2,2} + \Delta A_{2,3} + \Delta A_{2,4} + \ldots + \Delta A_{2,A-2} \\ \Delta A_{2,1} + \Delta A_{2,2} + \Delta A_{2,3} + \Delta A_{2,4} + \ldots + \Delta A_{2,A-3} \\ \Delta A_{2,1} + \Delta A_{2,2} + \Delta A_{2,3} + \Delta A_{2,4} + \ldots + \Delta A_{2,A-4} \\ . \\ \Delta A_{2,1} + \Delta A_{2,2} + \Delta A_{2,3} + \Delta A_{2,4} \\ \Delta A_{2,1} + \Delta A_{2,2} + \Delta A_{2,3} \\ \Delta A_{2,1} + \Delta A_{2,2} \\ \Delta A_{2,1} \qquad + \qquad \Delta A_{1,1}(A-1)+1 \end{bmatrix} \tag{54}$$

In equation (54) every rows of terms can be equal to a triangular shape of non-progression or progression, this is dependent to value of $n$. So for $i$-th row of a triangular can be equal to equation (55) and none-progression terms that never can change to a progression represents by $NP$ and $i$-th subtract of between successive terms that a row of triangular represents by $ROW_{,i}$.

$$ROW_{,i} = \begin{bmatrix} \Delta A_{i+1,1} + \Delta A_{i+1,2} + \Delta A_{i+1,3} + \ldots + \Delta A_{i+1,A-i} \\ \Delta A_{i+1,1} + \Delta A_{i+1,2} + \Delta A_{i+1,3} + \ldots + \Delta A_{i+1,A-(i+1)} \\ \Delta A_{i+1,1} + \Delta A_{i+1,2} + \Delta A_{i+1,3} + \ldots + \Delta A_{i+1,A-(i+2)} \\ . \\ \Delta A_{i+1,1} + \Delta A_{i+1,2} + \Delta A_{i+1,3} + \Delta A_{i,4} \\ \Delta A_{i+1,1} + \Delta A_{i+1,2} + \Delta A_{i+1,3} \\ \Delta A_{i+1,1} + \Delta A_{i+1,2} \\ \Delta A_{i+1,1} \qquad + \qquad NP \end{bmatrix} \tag{55}$$

In fact, we investigate that sum of all terms in equation (53) can be equal with one or more other of subtraction of successive terms that its first term greater than the last term of equation (53) or can't. So all exponential integer numbers have its progression and for apprehend that several terms of this equation can transform to other terms, this procedure is similar to cube of integer numbers. In equation (53) we must reduce triangular progression from this equation and versus increase the terms of progression. In which sum of all $NP$ in equation (53) and one of $NP$ are significant, so all the first term of the first subtraction between successive terms of exponential integer numbers is an odd number in opposite the common difference of progression is an even number and $\Delta A_{1,1}(A\text{-}1)+1$ in equation (54) and $A$ must be an odd numbers when decrease from $A$ to amount of terms that increase term of progression and sum with one until divide to common difference. Because the feature of exponential integer numbers is that when $n$ (power) is an odd number the first term of this exponential integer numbers able that divide completely and when $n$ be an even number then the sum of twice of the first term of its progression able divide completely, for this reason that $NP$ convert to common difference which then extra $NP$ use for increase of other exponential integer number of its progression terms.

As well as all of the first term ($\Delta A_{1,1}$, $\Delta A_{2,1}$, $\Delta A_{3,1}$, …, $\Delta A_{(n-1),1}$) of triangular non-progression must divide completely to common difference that until transform to common difference so important that find counts of the first terms of non-progression. thus the counts of $\Delta A_{1,1}$ equal to counts of terms of its row, and counts of $\Delta A_{2,1}$ for the largest term of its row equal to its digit number. For other terms $\Delta A_{2,1}$ equal to one unite less than the successive terms and counts of $\Delta A_{3,1}$ for one term equal to sum of natural numbers that last number of natural numbers equal to its digit number on its row terms for example the last out of term of non-progression equal to 8 then the counts of $\Delta A_{3,1}$ equal to 1+2+3+4+5+6+7+8. And for other terms one digit less from the last out than the successive terms for example, for 7 terms of non-progression, counts of $\Delta A_{3,1}$ equal to 1+2+3+4+5+6+7. Thus for finding of other counts of the first term following the procedure of $\Delta A_{3,1}$.

We apprehend that when the counts of non-progression that never change to a progression and this non-progression obtain from the first terms of subtraction exponent integer numbers and first terms more than one part at this time impossible that can transform the extra of the first terms of non-progression to common difference. The other word remains $\varepsilon x$ similar to progression of cube of integer numbers. Otherwise $A^n$ build complex of equation (29) with difference of first term, common difference and part of non-progression term and can take factor $R_1$ from this complex equation. So when from $A^n$ reduce triangular of progression and non-progression and opposed increase terms of progression, actually that build progression which count of terms of this progression less than initial progression and equation $A^n=R_1[s+(n\text{-}1)]^3$ that obtain from exchange of value from one term to other term that then can take factor or can write this form $A^n=R_{a1}s^3+R_{b1}s^2+R_{c1}s+NP_1$ and must change to $A^n=R_am^3+R_bm^2+R_cm+NP$ and where $s>m$ and $R_a$, $R_b$, $R_c$ are coefficient that when obtain which $s$ convert to $m$ and dependent to progression. Therefore, we ascertain in progression of cube of integer numbers impossible that equation (29) convert to equation (52) where repeat this procedure. So cannot change sum of all terms in equation (53) to one or more of subtraction of successive terms that its first term greater than the last term of this equation and announce we can't modify synthesis of sequence of power-exponential integer numbers when power greater than 2.

# 5. Proof of the Beal's Conjecture

### *5.1. Situation 1 When A, B, C Are Coprime and X, Y, Z > 2*

For proof of Beal's Conjecture investigate two situations: when $A$, $B$, $C$ are coprime numbers and other when $A$, $B$, $C$ have a common prime factor. The first situation $A^X$ must convert to incomplete shape of $B^Y$ or create a series of subtraction exponential integer number from $X$-th exponential integer numbers that the first term of this series must greater than last term of subtraction exponential integer numbers of $B^Y$, so investigate this procedure by below equations.

$$B^Y=1+\Delta B_{1,1}+\Delta B_{1,2}+\Delta B_{1,3}+\Delta B_{1,4}+\ldots+\Delta B_{1,B-1} \tag{56}$$

$$A^X=\Delta B_{1,B}+\Delta B_{1,B+1}+\Delta B_{1,B+2}+\Delta B_{1,B+3}+\ldots+\Delta B_{1,B+i}\ 1+\Delta A_{1,1}+\Delta A_{1,2}+\Delta A_{1,3}+\Delta A_{1,4}+\ldots+\Delta A_{1,A-1} \tag{57}$$

By compare of equation (56) and (57) and when $Y<X$, two subject are very important. The one of counts terms of progression of exponential integer numbers and other series of progression that appear to shape of triangular. Thus that both (counts terms of progression of power-exponential integer numbers and series of progression has shape of triangular) are note equal concurrent in equation (57). The first status if assume that counts terms of progression equal and count terms of progression of $A^X$ denoted to $n$ then have:

$$A^X=R_1\left[n+(X-1)\right]^3=R_{a1}n^3+R_{b1}n^2+R_{c1}n+NP_1=R_an^3+R_bn^2+R_cn+NP \tag{58}$$

In equation (58) counts term of subtraction successive terms in $A^X$ that equal to $\Delta B_{1,B}$+ $\Delta B_{1,B+1}$+ $\Delta B_{1,B+2}$ +…+ $\Delta B_{1,B+i}$ with sum of subtraction successive terms in $B^Y$ must less than counts terms of $\Delta A_{1,1}$+ $\Delta A_{1,2}$ $\Delta A_{1,3}$+ $\Delta A_{1,4}$+…+ $\Delta A_{1,A\text{-}1}$ amount of difference $X$ and $Y$. In which appear $R_{a1}n^3+R_{b1}n^2+R_{c1}n+NP_1>R_an^3+R_bn^2+R_cn+NP$ because coefficient of $R_1$ that obtain from exchange of value between $R_{a1}n^3$, $R_{b1}n^2$, $R_{c1}n$ and $NP_1$ specify by counts of series of progression that has shape of triangular, first term of progression and common difference that where greater than triangular progression shape, first term and common difference of $R_an^3$+ $R_bn^2$+$R_cn$+$NP$. Thus repeat of triangular in $A^X$ more than $B^Y$ when appear progression in both exponential integer numbers, thus this situation impossible. The second situation if assume series of triangular shape of arithmetic progression equal and then can write equation (59),

$$A^X = R_1\left[n+(X-1)\right]^3 = R_{a1}n^3 + R_{b1}n^2 + R_{c1}n + NP_1 = R_a s^3 + R_b s^2 + R_c s + NP \tag{59}$$

In which $s>n$, $R_{a1}$ greater than $R_a$ and $R_{b1}$, $R_{c1}$, $NP_1$ have difference value than $R_b$, $R_c$, $NP$ respectively and difference between progression parts, small or great value than together is equal amount of several smaller of common difference, so difference between $NP_1$, $NP$ is not a common difference. Nevertheless, this equation impossible that equal in integer numbers and if find $A^X$, $B^Y$ to similar other subtraction exponential integer numbers the above procedure repeated. Also can proof other way by below method and consider two situations:

When $Cr_1 + TA^3$ on equation (43) that equal to $(B^{y_1})^{y_2}$, in other words that $Y$ is not a prime numbers and $A$, $B$ are a prime number then outcome Beal's Conjecture.

For the second cause consider $A^X+(Cr_1+TA^X)= Cr_1+(T+1)A^X$ in which if assume $A^X$ can transform to $aD^{x_1} + bH^{x_1}$ that where $H$ greater than $D$ and $Cr_1+TA^X$ that be equal to $B^Y$, then can write equation (60).

$$A^X + B^Y = B^Y + aD^{x_1} + bH^{x_1} \tag{60}$$

If can obtain a factor between $B^Y$ and $aD^{x_1}$ in equation (60) that can write $B^Y$ to $a_3B^{x_1}$ or $a_3L^{x_1}$ that both status are same solution in which we just consider first status and also between $B$ and $D$ is not any factor, then can write below equation.

$$A^X + B^Y = a_1(a_2D^{x_1} + a_3B^{x_1}) + bH^{x_1} \tag{61}$$

The equation (61) can write form of subtraction exponential integer numbers so this equation become:

$$A^X + B^Y = a_1\left[a_2(1+\Delta D_{1,1} + \Delta D_{1,2} + \ldots + \Delta D_{1,D-1}) + a_3(1+\Delta B_{1,1} + \Delta B_{1,2} + \ldots + \Delta B_{1,B-1})\right] + bH^{x_1} \tag{62}$$

In equation (62) $a_2(1+\Delta D_{1,1}+\Delta D_{1,2}+\Delta D_{1,3}+\ldots+\Delta D_{1,D-1})$ increase terms amount of difference between $H$ and $D$ by $a_3(1+\Delta B_{1,1}+\Delta B_{1,2}+\Delta B_{1,3}+\ldots+\Delta B_{1,B-1})$ then can write $(a_1+b)H^{x_1}$, now investigate that this equation impossible or no. Thus consider the arithmetic progression of $a_3B^{x_1}$ that increase one term of $a_2D^{x_1}$ in every rows progression and non-progression terms, this procedure use for cube of exponential integer numbers that can zero $\varepsilon x$. So for cube of integer numbers have:

$$a_3\left\{\frac{n^2\left[2a_1+(n-1)d\right]}{2} - \frac{n(n-1)}{2}\left[\frac{n\left[2a_1+(n-1)d\right]}{2} - \frac{(n-1)\left[2a_1+(n-2)d\right]}{2}\right] + \frac{n(n-1)(n-2)d}{6} + 7n + 8\right\} = a_2\left\{\frac{s\left[2a_1+(s-1)d\right]}{2} + 7\right\} \tag{63}$$

Equation (63) has no solution in integer numbers, thus if increase terms of $a_2D^{x_1}$ more than one, then can write bellow equation and this equation impossible in integer numbers.

$$(n+2)^3 = \frac{a_2}{a_3}\left\{\frac{s\left[2a_1+(s-1)d\right]}{2} + \frac{(s+1)\left[2a_1+sd\right]}{2} + \ldots + \frac{(s+m)\left[2a_1+(s+m-1)d\right]}{2} + 7(m+1)\right\} \tag{64}$$

For proof of other exponential integer numbers that power greater than 3 has complex of equation (63) with difference of first term of progression, common difference and part of non-progression that similar to cube of integer numbers. For other word, exponent that greater than 3 cannot convert all $NP$ of exponential integer numbers to common difference of progression and remain $\varepsilon x$. Thus the Beal's Conjecture when $A$, $B$, $C$ are coprime numbers a form of Fermat's Last Theorem that illustrated by equation (61) and we deduce from this equation that compare of two integer number that power of greater than 2 and both number has not factor then no solution. In which can't increase terms of arithmetic progression of exponential integer number by other progression that outcome from other exponential integer numbers which both power is similar. In fact, (FLT) has direct relationship with Beal's Conjecture when $A$, $B$, $C$ are coprime numbers.

### *5.2. Situation 2 When A, B, C Have Coprime Factor*

The second situation of Beal's Conjecture that $A$, $B$, $C$ have common prime numbers are compare of equation (5) with equation (60), the essentially of the second situation is right triangle with a product of integer number. The right triangle one leg is an integer number and two legs are irrational numbers or two legs are integer numbers. So can write below equation,

$$A^X + B^Y = a_1\left\{a_2\left[\frac{n\left[2a_1+(n-1)d\right]+2}{2}\right] + a_3\left[\frac{m\left[2a_1+(m-1)d\right]+2}{2}\right]\right\} + bH^{y_1} \tag{65}$$

In equation (65) demonstrate that possible increase progression terms by other progression, so this equation has solution in integer numbers, thus the Beal's Conjecture outcome from primary triple that have not common prime factors. So investigate the triple that have not common prime factor, the relationship of right triangle is principle of Pythagoras theorem that can find relationship between Pythagoras theorem and right triangle. Resulting in,

We pose $2kc+k^2=2kc+k^2 \quad 2kc^3+k^2c^2=2ka^2c+k^2a^2+2kb^2c+k^2b^2 \Rightarrow$

$$\frac{2kc^3}{c^2}+\frac{k^2c^2}{c^2}=\frac{2ka^2c}{c^2}+\frac{k^2a^2}{c^2}+\frac{2kb^2c}{c^2}+\frac{k^2b^2}{c^2} \Rightarrow 2kc+k^2=\frac{2ka^2}{c}+\frac{k^2a^2}{c^2}+\frac{2kb^2}{c}+\frac{k^2b^2}{c^2} \Rightarrow$$

$$c^2+2kc+k^2=(a^2+b^2)+\frac{2ka^2}{c}+\frac{k^2a^2}{c^2}+\frac{2kb^2}{c}+\frac{k^2b^2}{c^2} \quad (c\pm k)^2=\left(a\pm\frac{ka}{c}\right)^2+\left(b\pm\frac{kb}{c}\right)^2 \tag{66}$$

The equation (66) determine the relationship between Pythagoras theorem and right triangle, in which $k$ contain in integer numbers if assume equation (66) to a circle then $k$ decrease or increase circle of radius and where $a$, $b$, $c$ are legs of right triangle that point $P_1$ moving to point $P_2$, showing in figure 5.

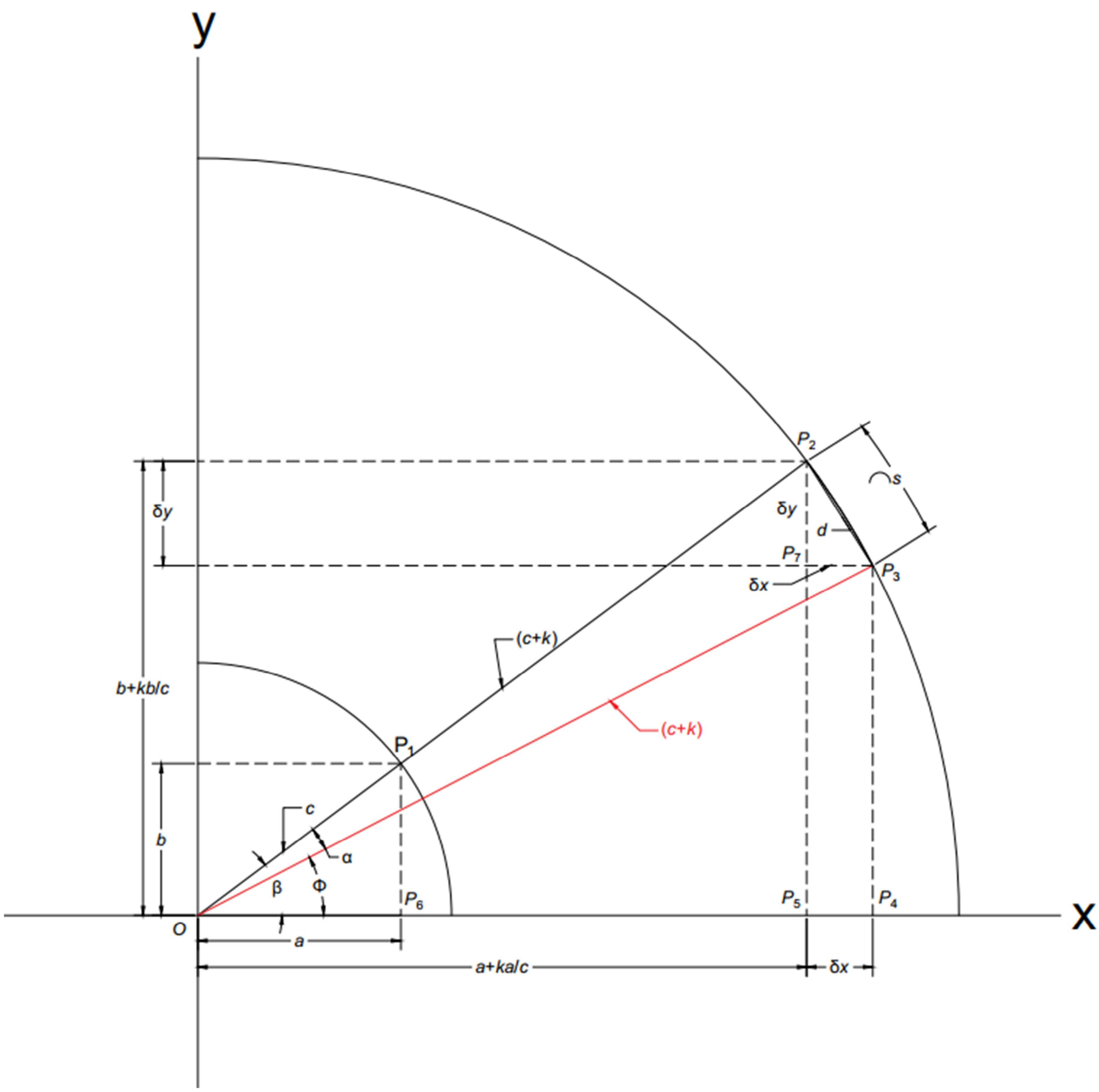


***Figure 5.*** *Graphic representation increase or decrease of circle of radius and displacement of points on this circle. Source: author.*

In figure 5 graphically represents when point $P_2$ displacement to point $P_3$ and investigate the primary triple of Beal's Conjecture, according to this figure and right triangle $P_2P_3P_7$ can exist equation (67) and (68),

$$d^2 = \delta x^2 + \delta y^2 \tag{67}$$

$$d = 2(c+k)\cdot \sin\left(\frac{\alpha}{2}\right) \tag{68}$$

$$(c \pm k)^2 = \left(a \pm \frac{ka}{c} + \delta x\right)^2 + \left(b \pm \frac{kb}{c} - \delta y\right)^2 \tag{69}$$

If combine equation (67), (68) and (69) will be appear below equations,

$$(c \pm k)^2 = \left\{a \pm \frac{ka}{c} + \sqrt{\left[2(c+k)\cdot \sin\left(\frac{\alpha}{2}\right)\right]^2 - \delta y^2}\right\}^2 + \left(b \pm \frac{kb}{c} - \delta y\right)^2 \tag{70}$$

### 5.2.1. Analysis of Right Triangle with Progression When Z, X or Z, Y Equal to 2

Equation (70) determine that all triple which has or has not common prime factor exist from Pythagoras theorem, so the triple of $C^2 = A^X \pm B^Y$ compare with equation (7), (29) and when $X$ or $Y$ equal to two then can write equation (71),

$$\frac{n\left[2a_1 + (n-1)d\right] + 2}{2} = \frac{s\left[2a_1 + (s-1)d\right] + 2}{2} \pm B^Y \tag{71}$$

And can write other form of equation (71) to equation (72),

$$n = \sqrt{(s+1)^2 - B^Y} - 1 \tag{72}$$

In equation (72) $2s + 1$ that equal to $B^Y$ has infinite solution, and when subtract amount of $p$ value from $s$ then equation (72) become to equation (73),

$$n = \sqrt{(s-p)^2 + 2s(p+1) - (p^2 - 1) - B^Y} - 1 \tag{73}$$

***Table 1.*** *Solutions for equation (73) and (72) that based are arithmetic progression. Source: author.*

| number | p | s | n | A | B | C | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 13 | 12 | 13 | 3 | 14 | 2 | 3 | 2 |
| 2 | 0 | 40 | 39 | 40 | 3 | 41 | 2 | 4 | 2 |
| 3 | 1 | 16 | 14 | 15 | 4 | 17 | 2 | 3 | 2 |
| 4 | 1 | 54 | 52 | 53 | 6 | 55 | 2 | 3 | 2 |
| 5 | 1 | 65536 | 65534 | 65535 | 8 | 65537 | 2 | 6 | 2 |
| 6 | 3 | 28 | 24 | 5 | 6 | 29 | 4 | 3 | 2 |

The equation (73) infinite integer numbers are exist which this equation satisfies and has not common prime factor, and also this equation has other form that two terms of triple have square power add with together and equal to an integer number which its exponent greater than two. So can write this equation to form of a progression. Resulting in,

$$\frac{n\left[2a_1 + (n-1)d\right] + 2}{2} + \frac{s\left[2a_1 + (s-1)d\right] + 2}{2} = B^Y \tag{74}$$

### 5.2.2. Analysis of Right Triangle with Progression When Z and Y >2

But for Z, Y > 2 must compare $A^2 \pm Cr + TA^2 = Cr + (T \pm 1)A^2$ in which $Cr + TA^2 = B^Y$ and if that can write $B^Y$ to form of $aD^2 + bH^2$ and $A^2$ to form of $a_3L^2$ or remain steady its form of $A^2$ in equation (75). thus can write below equations for $a_3L^2$,

$$A^2 \pm B^Y = a_1(a_2D^2 + a_3L^2) + bH^2 \tag{75}$$

$$A^2 \pm B^Y = a_1\left\{a_2\left[\frac{n\left[2a_1 + (n-1)d\right] + 2}{2}\right] + a_3\left[\frac{m\left[2a_1 + (m-1)d\right] + 2}{2}\right]\right\} + bH^2 \tag{76}$$

In equation (75), (76) counts terms of arithmetic progression of $D^2$, $L^2$ equal to $n$, $m$ respectively then progression of $a_3L^2$ increase terms of $a_2D^2$ amount of difference between $H$ and $D$ then can take factor and can write $(a_1+b)H^2$, in which this equation has infinite solution when triple has not any common prime factor. In addition, the equation (76) demonstrate that triple of $C^Z = A^2 \pm B^Y$ outcome from triple of Pythagorean theorem that retain its right angle. Thus equation $C^Z = A^2 \pm B^Y$ has solution infinite in integer numbers, so this equation product with $W$ number then outcome Beal's Conjecture according to figure 6.

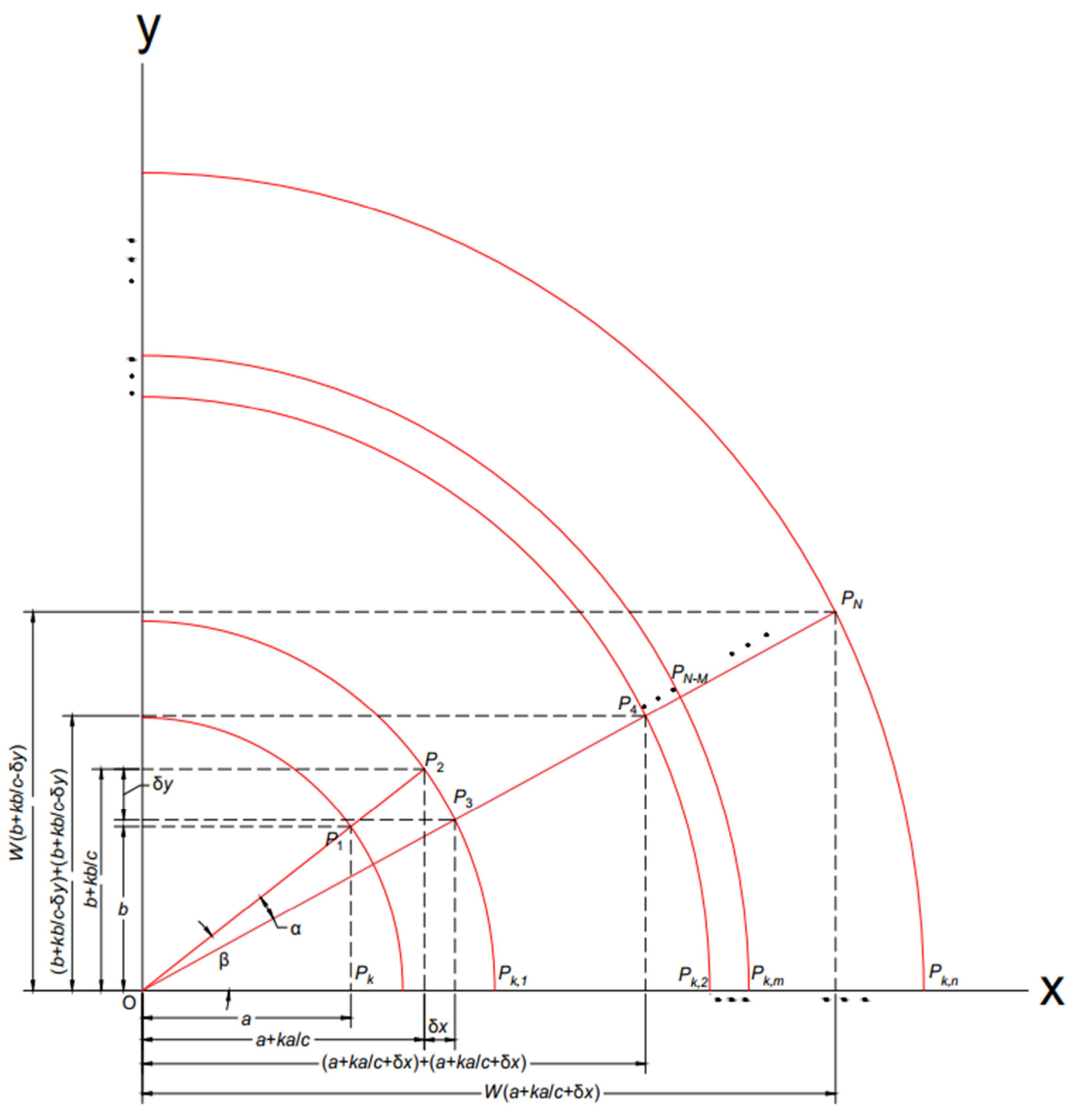


***Figure 6.*** *Graphic representation the Beal's circles that obtained from primary circles. Source: author.*

#### *5.2.3. Dependence of Beal's Conjecture to Fermat-Catalan's Equation*

The equation $C^2=A^X\pm B^Y$ two statuses may occur correspondent to equation (61) and (76):

The $m$ terms of progression of $B$ can increase terms of $D$ then can transform equation $c^2 = s^{x_1} \pm p^{x_2}$ to Beal's Conjecture. So that illustrated in figure 6 when point $P_2$ displace to point $P_3$, Pythagoras' equation change to Fermat-Catalan's equation but we apperceive Beal's Conjecture in super circle from primary circle. When $(c+k)$ radius increase to amount of $W$, if $W$ equal to $c^x$ then can write below equation.

$$c^{x+2} = c^x s^{x_1} \pm c^x p^{x_2} \tag{77}$$

In equation (77) $x$ can equal to product of $x_1$ and $x_2$ or equal to $x_1$ when $x_1$ be a factor of $x_2$. Thus $c^{x+2}$, $c^x s^{x_1}$ and $c^x p^{x_2}$ has a common factor $c$, if $c$ be a prime number then equation (77) has only one common prime factor otherwise $c$ a composite number we can $c$ decomposed represented prime factors. According to hypothesis, the prime factorization of a number is that product of prime numbers that equal the number or when a number is factored so that all its factors are prime numbers [9, 21].

Once the primes are infinite and $W$ an integer number by hypothesis, one can necessarily write $W$ in the form of the product of the infinite primes raised to their respective powers [1]. As we can write $W$ integer number to form the notation of infinite product.

$$W = \prod_{i=1}^{\infty} P_i^{K_{,i}} = P_1^{K_{,1}}.P_2^{K_{,2}}.P_3^{K_{,3}}...P_\infty^{K_{,\infty}} \tag{78}$$

In which $K_{,i}$ represents the number of times the $i$-$th$ prime appears in the factorization of $W$. Writing $c^{x+2}$, $c^x s^{x_1}$ and $c^x p^{x_2}$ in infinite product notation, one can have:

$$\left(\prod_{i=1}^{\infty} P_i^{(x+2)K_{,i}}\right) = \left(\prod_{i=1}^{\infty} P_i^{K_{,i}}\right)^x (s)^{x_1} \pm \left(\prod_{i=1}^{\infty} P_i^{K_{,i}}\right)^x (p)^{x_2} \tag{79}$$

$$P_1^{(x+2)K_{,1}}.P_2^{(x+2)K_{,2}}...P_\infty^{(x+2)K_{,\infty}} = \left(P_1^{K_{,1}}.P_2^{K_{,2}}.P_3^{K_{,3}}...P_\infty^{K_{,\infty}}\right)^x$$

$$(s)^{x_1} \pm \left(P_1^{K_{,1}}.P_2^{K_{,2}}.P_3^{K_{,3}}...P_\infty^{K_{,\infty}}\right)^x (p)^{x_2} \quad (80)$$

According to equation (61) $B^{x_1}$ impossible that increase terms of $D^{x_1}$ and does not create a progression then transform equation (77) to Beal's Conjecture. resulting in,

$$c^{x+1} = c^x s^{x_1} \pm c^x p^{x_2} \quad (81)$$

The equation (77) and (81) represents that both equation have common prime factors, if $c$ be a composed number. And also Beal's equation determines super circles that exist from primary circles, this primary circles buildup by Fermat-Catalan equation when $m$=2 or $m$=1 in which $m$ is denoted power of $c$. The below table show some solutions of Beal's Conjecture.

***Table 2.*** *Solutions for equation (81) and (77) when A, B, C have common prime factor. Source: author.*

| Common | $c^m$ | $s^{x_1}$ | $p^{x_2}$ | C | A | B | Z | X | Y |
|---|---|---|---|---|---|---|---|---|---|
| 29 | $29^2$ | $5^4$ | $6^3$ | 29 | 121945 | 4243686 | 14 | 4 | 3 |
| 71 | $71^2$ | $2^7$ | $17^3$ | 71 | 715822 | 154617042692647 | 23 | 7 | 3 |
| 2 and 19 | $76^2$ | $7^4$ | $15^3$ | 76 | 3072832 | 500432640 | 14 | 4 | 3 |
| 13 | $13^2$ | $8^3$ | $-7^3$ | 104 | 91 | 13 | 3 | 3 | 5 |
| 89 | 89 | $3^4$ | $2^3$ | 89 | 2114907 | 125484482 | 13 | 4 | 3 |
| 2417 | 2417 | $7^4$ | $2^4$ | 2417 | 238893669618247 | 68255334176642 | 17 | 4 | 4 |
| 359 | 359 | $7^3$ | $2^4$ | 359 | 116272185127 | 92536558 | 13 | 3 | 4 |

### *5.2.4. Dependence of Beal's Conjecture to Catalan's Equation and Some Other Situations*

The Beal's Conjecture has some special cases that these cases can create a circle and circle has some specific point that can outcome Beal's Conjecture according to equation (70) these primary circles attain from Pythagorean triple by increase or decrease of legs of right triangle and then displacement of points, the graphically illustrated in figure 6 so suppose the below equation,

$$c^Z = c^{Z-m} + c^{Z-m}(c^m - 1) \quad (82)$$

In equation (82) factor $c^m - 1^m$ a binomial quantity if that factorization, so then have,

$$(c-1)(c^{m-1} + c^{m-2} + c^{m-3} + c^{m-4} + ... + 1) = c^m - 1^m \quad (83)$$

when a sequence has a constant ratio between successive terms it is called a geometric progression. The constant is called common ratio [19]. If the first term of a GP is $a_1$ and the common ratio is $c$, then The $m$-th term is: $a_1 c^{m-1}$ and sum of $m$ terms of a GP equal to equation (84),

$$S_m = \frac{a_1(c^m - 1)}{(c-1)} \quad (84)$$

let that term $(c^{m-1} + c^{m-2} + c^{m-3} + c^{m-4} + ... + 1)$ of equation (83) is a geometric progression, thus product of parentheses of this equation equal to $b_1$ and power of $b_1$ equal to $Z - m$ that combine with $c^{Z-m}$. In accordance with Catalan's Theorem $3^2 - 1 = (3-1)(3+1)$, if compare Catalan's Theorem with equation (83) that apperceive this equation has only one solution when $m \geq 2$, and we find that: $S_m = \frac{a_1(c^m - 1)}{(c-1)} \neq b_1^{Z-m-1}$ for one parentheses and exception to Catalan's Theorem. But equation (83) has infinite solution when $m = 1$, thus this equation become for Catalan's Theorem:

$$(c-1) = b_1 \quad (85)$$

$$(c^{m-1} + c^{m-2} + c^{m-3} + c^{m-4} + ..... + 1) = b_1^{Z-m-1} \quad (86)$$

And equation (83) equivalent to equation (87), when $m = 1$

$$(c-1) = b_1^{Z-m} \quad (87)$$

let we can reform equation (82) to equation (88)

$$c^Z = c^{Z-m} + (cb_1)^{Z-m} \quad (88)$$

And where $c^{Z-m}$, $(cb_1)^{Z-m}$ Range of two legs of right triangle. So can write equation (88) to form of the product of the infinite primes raised to their respective powers. Resulting in,

$$P_1^{(Z)K_{,1}}.P_2^{(Z)K_{,2}}...P_\infty^{(Z)K_{,\infty}} = P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}} + b_1^{Z-m}\left(P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}\right) \quad (89)$$

In equations (88) to be seen $c^Z$, $c^{Z-m}$ and $(cb_1)^{Z-m}$ has common primes factor, when $c$ composed number. According to figure 6 and $W$ equal to equation (90),

$$W = P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}} \quad (90)$$

If in equation (89) we cancel primes factors. Then apprehend that:

$$\frac{P_1^{(Z)K_{,1}}.P_2^{(Z)K_{,2}}...P_\infty^{(Z)K_{,\infty}}}{P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}} = \frac{P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}}{P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}} + \frac{b_1^{Z-m}\left(P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}\right)}{P_1^{(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}}} \Rightarrow$$

$$P_1^{(Z)K_{,1}-(Z-m)K_{,1}}.P_2^{(Z)K_{,2}-(Z-m)K_{,2}}...P_\infty^{(Z)K_{,\infty}-(Z-m)K_{,\infty}} = P_1^{(Z-m)K_{,1}-(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}-(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}-(Z-m)K_{,\infty}}$$

$$+b_1^{Z-m}\left(P_1^{(Z-m)K_{,1}-(Z-m)K_{,1}}.P_2^{(Z-m)K_{,2}-(Z-m)K_{,2}}...P_\infty^{(Z-m)K_{,\infty}-(Z-m)K_{,\infty}}\right) \Rightarrow P_1^{(m)K_{,1}}.P_2^{(m)K_{,2}}...P_\infty^{(m)K_{,\infty}} = 1+b_1^{Z-m}(1) \quad c^m = 1+b_1^{Z-m} \tag{91}$$

Equation (91) equivalent to equation (70) when $\left(b+\frac{kb}{c}-\delta y\right)=1$, then in equation (91) five situation may occur:

When $m = 2$, $Z - m = 3$ showed Catalan's equation and a right triangle. Represented in equation (92).

$$\left(\sqrt{c^m}\right)^2 = \left(\sqrt{1}\right)^2 + \left(\sqrt{b_1^{Z-m}}\right)^2 \tag{92}$$

For $m$=1 and $Z–m$≥3 rewriting equation (91) to equation (93), this equation also specifies right triangles.

$$\left(\sqrt{c}\right)^2 = \left(\sqrt{1}\right)^2 + \left(\sqrt{b_1^{Z-m}}\right)^2 \tag{93}$$

When $m$≥3 and $Z–m$=1 this situation also attained by increase and decrease of hypotenuse of right triangle and movement of points.

$$\left(\sqrt{c^m}\right)^2 = \left(\sqrt{1}\right)^2 + \left(\sqrt{b_1}\right)^2 \tag{94}$$

In situation four when $b_1$=1 then equation (91) reform to equation (95), as also can find Beal's Conjecture.

$$\left(\sqrt{2}\right)^2 = \left(\sqrt{1}\right)^2 + \left(\sqrt{1}\right)^2 \tag{95}$$

When $m$=1, $Z–m$=1 this status is also right triangle but is not subtend to Beal's Conjecture. Represented in equation (96).

$$\left(\sqrt{c}\right)^2 = \left(\sqrt{1}\right)^2 + \left(\sqrt{b_1}\right)^2 \tag{96}$$

There are some solutions of Beal's Conjecture in table 3.

***Table 3.*** *Solutions for equations (92), (93), (94) and (95) which those are based Pythagoras triple. Source: author.*

| Common | c | Z-m | $b_1$ | C | A | B | Z | X | Y |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 2 | 3 | 6 | 3 | 5 | 3 | 3 |
| 3 | 3 | 6 | 2 | 3 | 18 | 3 | 8 | 3 | 6 |
| 17 | 17 | 4 | 2 | 17 | 34 | 17 | 5 | 4 | 4 |
| 257 | 257 | 8 | 2 | 257 | 514 | 275 | 9 | 8 | 8 |
| 7 | 2 | 0 | 7 | 14 | 7 | 7 | 3 | 4 | 3 |
| 31 | 2 | 0 | 31 | 62 | 31 | 31 | 5 | 6 | 5 |
| 2 | 2 | 0 | 1 | 2 | 2 | 2 | 4 | 3 | 3 |
| 2 | 2 | 0 | 1 | 16 | 2 | 2 | 4 | 15 | 15 |
| 3 and 11 | 33 | 5 | 2 | 33 | 66 | 33 | 6 | 5 | 5 |

# 6. Conclusion

The first term, distinguished exponent integer numbers create arithmetic progression when subtract successive terms, and square of integer numbers has a row of progression that can obtain Pythagoras triple by this progression. The cube of integer number makes several series of arithmetic progression that has a shape of triangular.

As the second output, the demonstration clarified the relationship between Catalan's Theorem and arithmetic progression and represented that equation $n=\sqrt[3]{m^3\left(\frac{1}{m}+\frac{2}{m^2}\right)}-2$ has only one solution in integer numbers. In situation two $n=\sqrt[3]{m^3\left(\frac{1}{m}+\frac{2}{m^2}+\frac{2}{m^3}\right)}-2$ impossible which this equation has a complete root in integer numbers and for $p$>3, $q$=2 or $q$>3, $p$=2, then its proof of similar to before situation. The last status demonstrated that ($p$, $q$)-th power of integer numbers can't an arithmetic progression without 1 that is the synthesis of progression of exponent integer numbers which is exist value of one.

The third part proved the Fermat's Last theorem, by increase of term of progression and illustrated that impossible disarrangement of synthesis of $n$-the exponent integer numbers to a row of arithmetic sequence or more row of progression than one.

And the last part, presented that the Beal's Conjecture when X, Y, Z>2 and A, B, C has not common prime factor also dealt with several series of arithmetic progression that shape of this progression is similar to triangular and difference between (FLT) and Beal's Conjecture is coefficient. This coefficient product with all series of progression and non-progression parts and impossible increase terms of progression by other progression of subtraction successive terms of exponential integer numbers, the other word Beal's Conjecture a form of Fermat's Last Theorem. The second situation of Beal's Conjecture when A, B, C has common prime factor dealt with one series of arithmetic progression and this status make circles and these circles are super circles that coming into existence from primary circles. This primary circles created from Diophantine equations that consist of $c^m=1+b_1^{Z-m}$ with four situations ($m$=2, $Z–m$=3) Catalan equation, ($m$ ≥ 3 and $Z–m$=1), ($m$=1 and $Z–m$ ≥ 3), (when $b_1$=1) and Fermat-Catalan equation when $m$=2 or $m$=1.

# References

[1] Di Gregorio, L, T. (2013). Proof for the Beal conjecture and a new proof for Fermat's last theorem. Pure and Applied Mathematics Journal. Vol. 2, No. 5, pp. 149-155.

[2] Aithani, H, S. (218). The Mathematical Solution for the Beal Conjecture. AMIERJ. Volume- VII, Issues- IV, ISSN-2278-5655.

[3] Buya, S, B. (2019). A Review of Buya's Proof of Beal's Conjecture and Simple Proof of Fermat's Last Theorem. American Journal of Mathematical Analysis. Vol. 7, No. 1, 15-16. DOI: 10.12691/ajma-7-1-3.

[4] Buya, S, B. (2021). A final of Buya's Proof of Beal's Conjecture. ResearchGate.

[5] Nyambuya, G, G. (2014). A Simple and General Proof of Beal's Conjecture. Advances in Pure Mathematics. 4, 518-521.

[6] Buya, S, B. (2018). Proof of Beal's conjecture. African Journal of Mathematics and Computer Science Research. Vol. 11 (8), PP. 109-113. ISSN: 2006-9731. DOI: 10.5897/AJMCSR2018.0764.

[7] Porras Ferreira, J, W. (2014). The Solution for Fermat's Last Theorem using Exponential Algebra Properties. Global Open Research Journal of Mathematics. Vol. 2, No. 1, PP. 01-11.

[8] Stewart, I., & Tall, D. (2002). Algebraic Number Theory and Fermat's Last Theorem. A K Peters, Ltd. 63 South Avenue. Natick, MA 01760. ($3^{th}$ ed).

[9] Marecek, L., Anthony-Smith, M, A,. & Mathis, A, H. (2020). Elementary Algebra 2e. Rice University. 6100 Main Street MS-375, Houston, Texas 77005.

[10] Di Gregorio, L, T. (2018). A Deeper Analysis on a Generalization of Fermat's Last Theorem. Canadian center of Science and Education. Journal of Mathematics Research. Vol. 10, No. 2. ISSN 1916-9795, E-ISSN 1916-9809.

[11] Mazaris, A. (2018). A new Proof of Fermat's Last theorem. Open Access Journal of Mathematical and Theoretical Physics. Volume 1, Issue 3.

[12] Kumar, V. (2014). Proof of Fermat Last Theorem based on Odd Even Classification of integers. Int. J. Open Problems Compt. Math. Vol. 7, No. 4. ISSN 1998-6262.

[13] Sghiar, M. (2020). The Fermat Classes and The Proof of Beal Conjecture. IOSR Journal of Mathematics (IOSR-JM). Volume 16. Issue 3. PP 09-13. e-ISSN: 2278-5728, p-ISSN: 2319-765X.

[14] Frey, G. (2009). The Way to the Proof of Fermat's Last Theorem. DE TOULOUSE Mathématiques. Vol. XVIII, PP. 5-23.

[15] Metsankyla, T. (2003). Catalan's Conjecture: Another Old Diophantine Problem Solved. Bulletin (New Series) of the American Mathematical Society. Volume 41, Number 1, Pages 43-57.

[16] Koymans, P, H. (2017). The Catalan equation. Indagationes Mathematicae. 321-352.

[17] Daems, J. (2003). A Cyclotomic Proof of Catalan's Conjecture.

[18] Darmon, H., & Granville, A. (1995). On The Equations $z^m=F(x,y)$ And $Ax^p+By^q=Cz^r$. Bull. London Math. Soc. 27 (1995) 513-543.

[19] Bird, J. (2017). Higher Engineering Mathematics. Routledge, 2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN. 711 Third Avenue, New York, NY 10017. ($8^{th}$ ed).

[20] Lemma, M., Lambright, J., & Epps, B. (2015). The Mathematical Beauty of Triangular Numbers. Hawaii University International Conferences. ISSN 2333-4916 (CD-ROM), ISSN 2333-4908 (ONLINE).

[21] Ellis, W., & Burzynski, D. (2012). Elementary Algebra. Rice University, Houston, Texas.