International Journal of Materials Science and Applications
2022; 11(2): 42-47
http://www.sciencepublishinggroup.com/j/ijmsa
doi: 10.11648/j.ijmsa.20221102.11
ISSN: 2327-2635 (Print); ISSN: 2327-2643 (Online)

Chiral Waves and Topological Novel States in Fermi

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To cite this article:

Subhamoy Singha Roy. Chiral Waves and Topological Novel States in Fermi. *International Journal of Materials Science and Applications.* Vol. 11, No. 2, 2022, pp. 42-47. doi: 10.11648/j.ijmsa.20221102.11

Received: December 31, 2021; **Accepted**: January 17, 2021; **Published**: March 12, 2021

Abstract: This first show was that the spin system, the two nearest neighbour spin in the Entanglement of measuring compliance by the ground conditions are achieved Berry stage to the relation when it is a closed path developed is. It noted the need to share the stage in a kind of geometric phase as explained to go to which any deformations involved is not and it's the first Stiefel-Whitney class which is to Z_z–cohomology value takes. However for polarized fermions we can relate the exchange phase as the celebrated Berry phase as in this case the Z_z-cohomology becomes irrelevant and is consistent with the first Chern class which involves curvature. This follows from the depiction of a fermion as a scalar particle attached with a magnetic flux line. As in this framework the measure of entanglement of two nearest neighbor spins in a spin system given by concurrence is found to be associated with the Berry phase acquired by a spin state when it evolves in a closed path we can consider that entanglement is a consequence of Fermi statistics. It has been noted that in terms of quantum field theory, the berry phase is related to the perpetual inconsistency caused by the breaking of the perpetual symmetry. As mentioned earlier the quantization procedure of a fermion in the framework of Nelson's stochastic quantization procedure introduces an internal variable which appears as a direction vector and gives rise to spin degrees of freedom. It's pointing to is that when a spin- 1 state of two spin 1/2 state is a Entangle system as considered to be that, the most widely covered state longitudinal elements with the matching it.

Keywords: Berry Phase, Entanglement, Chiral Anomaly, Quantum Field Theory

1. Introduction

Entanglement is a specific feature of quantum mechanics. It is known that quantum mechanics is nonlocal and exhibits a peculiar correlation between two physically distant parts of the total system. Indeed it has been observed that the Bell inequality (BI) [1] can be violated in quantum mechanics but has to be satisfied by all local realistic theories. The violation of BI demonstrates the presence of entanglement. It is expected that this specific feature of quantum mechanics is induced through the quantization procedure. Klauder [2] has pointed out that quantization can be achieved in terms of a universal magnetic field acting on a free particle moving in a higher dimensional space when quantization corresponds to freezing the particle in its lowest Landau level. The significant aspect of this quantization procedure is that it has the specific property of coordinate independence and is governed by geometry. It has been pointed out that this formulation is equivalent to the geometric quantization [3] where the Hermitian line bundle takes a significant role. Also it has been shown that this procedure has its relevance in the quantization of a fermion [4, 5] in the framework of Nelson's stochastic quantization procedure [6] when a spinning particle is endowed with an internal degree of freedom through a direction vector (vortex line) which is topologically equivalent to a magnetic flux line. In view of this specific feature of the role of magnetic field in all these formulations of quantization procedure it is expected that the peculiar property of entanglement in quantum mechanics has its relevance with the magnetic flux associated with the quantization procedure. In a seminal paper Berry [7] has shown that when a quantum particle moves in a closed path in a parameter space it attains a geometric phase apart from the dynamical phase. The geometric phase belongs to the Chern class and corresponds to the holonomy [8]. It is given by the integral over the closed path of the relevant gauge potential. Essentially this phase is proportional to the number of magnetic flux lines enclosed by the path. In some recent papers [9-11] it has been shown that the measure of entanglement given by concurrence for a bipartite system of two nearest neighbor spins in a spin system effectively corresponds to the Berry phase factor attained by the ground state when the system is rotated in a closed path. It has also been shown that the quantification of spin entanglement in terms of Berry phase generalizes the relationship between the entanglement of distinguishable spins and that of delocalized identical fermions [11].

In sec. 2 we shall briefly review the relationship between entanglement, Berry phase and chiral anomaly. In sec. 3 we shall consider the Berry phase in an entangled state and the role of renormalization group (RG) flow associated with this.

2. Entanglement, Berry Phase and Chiral Anomaly

In some recent papers [9-11] it has been pointed out that in a spin system the measure of entanglement for two nearest neighbor spins given by concurrence effectively corresponds to the Berry phase factor acquired by a spin state when each spin is rotated about the quantization (z-) axis. In fact when a fermion is depicted as a scalar particle attached with a magnetic flux line quantum entanglement of two nearest neighbor spins in a spin system can be visualized as to be caused by the deviation of the internal magnetic flux line attached with one particle in presence of the other. This helps us to consider the measure of entanglement given by concurrence in terms of the Berry phase acquired by a spin state arising due to the rotation of the spin around the z-axis induced by the internal magnetic field associated with the other particle. It has also been shown that the spatial entanglement between two identical fermions at different spatial regions is associated with the corresponding spin entanglement [11]. In a recent paper [15] it has been shown that the Pauli sign of the spin-statistics relation of fermions essentially corresponds to this geometric phase and given by concurrence is found to be associated with the Berry phase acquired by a spin state when it evolves in a closed path we can consider that entanglement is a consequence of Fermi statistics [16, 17]. This helps us to consider spin as an $SU(2)$ gauge bundle [5, 12-15]. In fact we can now write the extended space-time coordinate as well as momentum as gauge covariant operator acting on functions in phase space

$$Q_\mu = -i(\frac{\partial}{\partial p_\mu} + A_\mu(p))$$

$$P_\mu = -i(\frac{\partial}{\partial q_\mu} + B_\mu(q)) \tag{1}$$

with the gauge field $A_\mu(B_\mu) \in SU_2$ and $q_\mu(p_\mu)$ denoting the mean position (momentum). It is observed that space-time coordinates as well as momentum variables given by eqn.(1) represent noncommutative geometry as their components do not commute. In fact the noncommutativity parameter is given by

$$[Q_\mu, Q_\nu] = F_{\mu\nu}(p)$$

$$[P_\mu, P_\nu] = F_{\mu\nu}(q) \tag{2}$$

where $F_{\mu\nu}$ represents the non-Abelian gauge field strength. The functional dependence of the noncommutativity parameter effectively corresponds to the existence of monopoles [18, 19]. This implies that the effect of monopoles is implicit in this formulation.

The angular momentum of a charged particle in the field of a magnetic monopole is given by:

$$\vec{J} = \vec{r} \times \vec{p} - \mu\hat{r} \tag{3}$$

where μ takes the value $\mu = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, ...$ This suggests that a fermion can be viewed as a scalar particle having orbital momentum 1/2 in this space. As $\mu = \frac{1}{2}$ corresponds to one magnetic flux line a fermion can be depicted as a scalar particle attached with a magnetic flux quantum. It may be mentioned that this formulation is analogous to the boson-fermion transformation in (2+1) dimensions which is achieved through the introduction of the Chern-Simons field. In 3-dimensional manifold the non-Abelian Chern-Simons action is given by

$$S(A) = \frac{k}{8\pi} \int_{M_3} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{4}$$

where k is a integer and A is the one-form associated with the Chern-Simons gauge field. The Chern-Simons invariant given by eqn.(4) is related to the Pontryagin index through the relation

$$\int_{M_4} Tr(F \wedge F) = \int_{M_3} Tr(A \wedge dA + (2/3) A \wedge A \wedge A) \tag{5}$$

where F is the two-form related to the field strength. The Pontryagin index q is associated with this 4-dimensional integral and is related to the magnetic monopole charge μ through the relation $q = 2\mu$. Noting that SU(2) is the covering group of SO(3) and $SL(2,C)$ is the covering group of the Lorentz group SO(3,1) we can generalize the non-Abelian gauge field A_μ as $SL(2,C)$) gauge field. In deed the generators of the $SL(2,C)$ group in the tangent space are given by:

$$g^1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; g^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, g^3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6}$$

We consider the topological Lagrangian in terms of the $SL(2,C)$ gauge fields in affine space

$$L = -(1/4) Tr\varepsilon^{\mu\nu\lambda a} F_{\mu\nu} F_{\lambda\sigma} \tag{7}$$

This gives rise to the topological current [20]

$$\vec{J}_\mu = \varepsilon^{\mu\nu\lambda a}\vec{a}_\nu \times \vec{F}_{\lambda\sigma} = \varepsilon^{\mu\nu\lambda a}\partial_\nu \vec{f}_{\lambda a} \tag{8}$$

where we have taken $SL(2,C)$ gauge fields

$$A_\mu = \vec{a}_\mu . \vec{g} \text{ and } F_{\mu\nu} = \vec{f}_{\mu\nu} . \vec{g} \tag{9}$$

It is noted that chiral anomaly appears when a chiral fermionic current interacts with a gauge field. The Lagrangian for the interaction of the Dirac field with the $SL(2,C)$ gauge field (neglecting the mass term) is given by

$$L = -\bar{\psi}\gamma_\mu D_\mu \psi - \frac{1}{4} Tr \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \tag{10}$$

where D_μ is the $SL(2,C)$ gauge covariant derivative given by $D_\mu \equiv \partial_\mu - igA_\mu$ where g is some coupling constant. Here D_μ is the gauge covariant derivative defined by $D_\mu \equiv \partial_\mu - igA_\mu$, g being the coupling constant. If we split the Dirac massless spinor in chiral form and identify the internal helicity +1/2 (-1/2) with left (right) chirality corresponding to $\psi_L(\psi_R)$ we can write

$$\begin{aligned}&\bar{\psi}\gamma_\mu D_\mu \psi = \bar{\psi}\gamma_\mu \partial_\mu \psi = ig\bar{\psi}\gamma_\mu A_\mu^a g^a \psi = \bar{\psi}\gamma_\mu \partial_\mu \psi \\ &-\frac{ig}{2}(\bar{\psi}_R \gamma_\mu A_\mu^1 \psi_R - \bar{\psi}_R \gamma_\mu A_\mu^2 \psi_R + \bar{\psi}_L \gamma_\mu A_\mu^2 \psi_L + \bar{\psi}_L \gamma_\mu A_\mu^3 \psi_L)\end{aligned} \tag{11}$$

This gives rise to the following three conservation laws [13]

$$\partial_\mu[\frac{1}{2}(-ig\bar{\psi}_R \gamma_\mu \psi_R) + J_\mu^1] = 0$$

$$\partial_\mu[\frac{1}{2}(-ig\bar{\psi}_L \gamma_\mu \psi_L + ig\bar{\psi}_R \gamma_\mu \psi_R) + J_\mu^2] = 0$$

$$\partial_\mu[\frac{1}{2}(-ig\bar{\psi}_L \gamma_\mu \psi_L) + J_\mu^3] = 0 \tag{12}$$

where $J_\mu^i (i = 1,2,3)$ are gauge field currents given by eqn.(8).

These three equations represent a consistent set of equations if we choose $J_\mu^1 = -\frac{1}{2} J_\mu^z$, $J_\mu^3 = +\frac{1}{2} J_\mu^z$. This guarantees the vector current conservation. From this we can write

$$\partial_\mu[\bar{\psi}_R \gamma_\mu \psi_R + J_\mu^2] = 0$$

$$\partial_\mu[\bar{\psi}_L \gamma_\mu \psi_L - J_\mu^2] = 0 \tag{13}$$

Thus we have

$$\partial_\mu(\bar{\psi}\gamma_\mu \gamma_5 \psi) = \partial_\mu J_\mu^5 = -2\partial_\mu J_\mu^2 \tag{14}$$

The chiral anomaly $\partial_\mu J_\mu^5 \neq 0$ is expressed here in terms of the gauge field current J_μ^2. The charge corresponding to this current is

$$q = \int J_0^2 d^3x = \int_{surface} \in^{ijk} d\sigma_i F_{jk}^2 \text{ , i,j,k=1,2,3} \tag{15}$$

Visualizing F_{jk}^2 to be the magnetic field like component for the vector potential A_μ^2 we see that q actually corresponds to the magnetic pole strength μ given by $q = 2\mu$. Thus we find that the chiral anomaly essentially is associated with the magnetic monopole charge. The fact that chiral anomaly is related to the Berry phase follows from this relationship of chiral anomaly with monopole charge. In fact the Berry phase factor is proportional to the number of magnetic flux lines enclosed by the loop traversed by the particle when it evolves in a closed path. It is noted that when the internal coordinate ξ_μ representing the direction vector is attached to the space-time point x_μ, for the complexified coordinate $z_\mu = x_\mu + i\xi_\mu$, we should take into account the polar coordinates *r, θ, φ* for space components of the vector x_μ and the angle χ to specify the rotational orientation of the direction vector ξ_μ. The eigenvalue of the operator $\frac{i\partial}{\partial\chi}$ corresponds to the monopole charge. Indeed the angle χ effectively takes care of the extension of the canonical system with certain internal structure. For compactified space this enlarges the configuration space $s^2 \to s^3$ and the angle χ acts like U(1) gauge degree of freedom that represents Hopf fibration of $s^2 \to s^3$. It is observed that a quantized Dirac monopole can be treated as the Hopf bundle U(1) over s^2 [20, 21]. The angular part of the spherical harmonics associated with the angle χ is given by $e^{-i\mu x}$ [12]. So from the relation

$$i\frac{\partial}{\partial\chi} e^{-i\mu x} = \mu e^{-i\mu x} \tag{16}$$

we note that when χ is changed to $\chi + \delta\chi$, we have the relation

$$i\frac{\partial}{\partial(\chi+\delta\chi)} e^{-i\mu x} = i\frac{\partial}{\partial(\chi+\delta\chi)} e^{-i\mu(\chi+\delta\chi)} e^{i\mu\delta\chi} \tag{17}$$

This implies that the wave function will acquire an extra factor $e^{i\mu\delta\chi}$ due to the infinitesimal change of the angle χ. For one complete rotation the phase is

$$e^{i\mu} \int_0^{2\pi} \delta\chi = e^{i2\pi\mu} \tag{18}$$

which is the Berry phase [12]. So from the relation

$$q = 2\mu = \int J_0^2 d^3x = \int \partial_\mu J_\mu^2 d^4x = -\frac{1}{2}\int_0^{2\pi} \partial_\mu J_\mu^5 d^4x \quad (19)$$

we note that the Berry phase is related to the integral of the anomaly. From this analysis it is now observed that the measure of entanglement for two nearest neighbor spins in a spin system given by concurrence which is shown to be related to the Berry phase factor acquired by the spin state when it evolves in closed path essentially relates it to the chiral anomaly which is generated when the associated chiral fermion interacts with a gauge field. In fact the change of the alignment of spin orientations caused by entanglement gives rise to this anomaly. Thus entanglement in a spin system corresponds to the effect of chiral symmetry breaking leading to chiral anomaly which is manifested through the Berry phase.

3. Entanglement, Berry Phase and Renormalization Group Flow

A general bipartite state can be written as

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \quad (20)$$

where *α, β, γ, δ* are complex coefficients satisfying the normalization condition. The state $|0\rangle$ and $|1\rangle$ correspond to the down and up spins respectively. The measure of entanglement viz. concurrence *C* is given by [22]

$$C = |2(\alpha\delta - \beta\gamma)| \quad (21)$$

As mentioned earlier we consider that under the influence of the magnetic flux line associated with one spin, the magnetic flux line of the other will deviate from the z-axis. We may view that the magnetic field is rotating with an angle *θ*. The time dependent magnetic field is given by

$$\vec{B}(t) = B\vec{n}(\theta, t) \quad (22)$$

Where $\vec{n}(\theta,t)$ is a unit vector which can be chosen as

$$\vec{n}(\theta,t) = \begin{pmatrix} sin\theta cos\omega_0 t \\ sin\theta \sin\omega_0 t \\ cos\theta \end{pmatrix}$$

The interaction is described by the Hamiltonian

$$H = \frac{k}{2} B\vec{n}.\alpha \quad (23)$$

where $\vec{\alpha}$ is the vector of Pauli matrices and $k = g\mu_B$, μ_B , being the Bohr magneton and *g* is the Lande factor. The instantaneous eigenstate of a spin operator in direction $\vec{n}(\theta,t)$ expanded in the α_z basis is given by

$$|\uparrow_n; t\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2} e^{i\omega_0 t}|\downarrow\rangle$$

$$|\downarrow_n; t\rangle = \sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2} e^{i\omega_0 t}|\downarrow\rangle \quad (24)$$

For the time evolution from $t = 0$ to $t = T$ where $T = \frac{2\pi}{\omega_0}$ each eigenstate will acquire a geometric phase apart from the dynamical phase. We can write

$$|\uparrow_n; t=0\rangle \to |\uparrow_n; t=T\rangle = e^{i\gamma+(\theta)} e^{i\nu+} |\uparrow_n; t=0\rangle$$

$$|\downarrow_n; t=0\rangle \to |\downarrow_n; t=T\rangle = e^{i\gamma-(\theta)} e^{i\nu-} |\downarrow_n; t=0\rangle \quad (25)$$

where $\gamma_\pm$ is the geometric phase and $\nu_\pm$ is the dynamical phase. The geometric phase is found to be given by [9-11]

$$\gamma_+(\theta) = -\pi(1-\cos\theta) \quad (26)$$

$$\gamma_-(\theta) = -\pi(1+\cos\theta) = -\gamma_+(\theta) - 2\pi \quad (27)$$

In this entangled state the angle *θ* corresponds to the deviation of the spin axis from the z-axis. It has been pointed out earlier that when a scalar particle moves in a closed path the geometric (Berry) phase is given by $e^{i2\pi\mu}$ where *μ* corresponds to the monopole charge. It is noted that *μ* = 1=2 corresponds to one magnetic flux line and when a scalar particle encircles one magnetic flux line the system generates a *π*-phase representing a fermion. In an entangled state of spin 1/2 systems we note that the Berry phase factor $\gamma/2\pi$ essentially corresponds to the monopole charge which we denote as $\tilde{\mu}$. Thus we find that the effective monopole charge associated with a spin state in an entangled system is given by

$$\tilde{\mu} = \frac{1}{2}(1-\cos\theta) \quad (28)$$

which essentially represents the concurrence. The entanglement content between two systems A and B in a pure state is measured by the entanglement entropy known as the Von Neumann entropy associated with the reduced density matrix ρ_A (equivalently ρ_B) given by

$$S = -Tr\rho_A \log_2 \rho_A \quad (29)$$

Osborne and Nielson [23] have pointed out that the entanglement of formation given by concurrence in a mixed state reduces to the Von Neumann entropy in a pure state. It has been shown by Casini and Huerta [24] that the entanglement entropy undergoes a renormalization group (RG) flow. Indeed Holzhey et. al. [25] have shown that for an one dimensional spin system undergoing quantum phase transition the entanglement entropy of a block of *L* spins with the rest of the system at criticality is proportional to the central charge *c* associated with the conformal field theory. In an earlier paper [26] it has been pointed out that the monopole charge associated with the Berry phase which is related to chiral anomaly has its correspondence with the central charge associated with the conformal anomaly in 1+1 dimensional conformal field theory. Zamolodchikov [27] has noted that the central charge c undergoes a RG flow. In view of this we note that the monopole charge μ associated with the Berry phase also undergoes a RG flow [28].

When μ depends on a certain parameter λ we have:

1. μ is stationary at fixed values λ^* of the RG flow. 2. at the fixed points $(\mu)\lambda^*$ is equal to the monopole charge given by quantized values $(\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots\ldots\ldots\ldots\ldots)$ 3. it decreases along the RG flow ie. $L\frac{\partial\mu}{\partial L} \le 0$ where L is a scale parameter. We now identify μ (λ) with $\tilde{\mu}$ in eqn.(28) with λ corresponding to θ. This suggests that for fixed values $\theta = 0, \pi/2$ and π we have the specific quantized values of the monopole charge $\tilde{\mu} = 0, 1/2$ and 1 respectively. $\tilde{\mu}$ decreases from the maximum value 1 at $\theta = \pi$ to 0 at $\theta = 0$ when entanglement entropy vanishes. It is noted that $\tilde{\mu} = \frac{1}{2}(\theta = \pi/2)$ is a specific value that gives a π-phase when the spin axis becomes orthogonal to the z-axis and spin-charge separation occurs [29]. For $\theta = \pi$ the spin state is reversed with $\tilde{\mu} = 1$ defining the maximally entangled state. For other values of θ, the effective monopole charge $\tilde{\mu}$ appears to be nonquantized. In fact this is essentially related to the RG flow associated with the entangled state.

It is noted that at $\theta = 0$ the entanglement entropy vanishes and we have the product state. From this analysis we observe that in this case the Berry phase is trivial with $\tilde{\mu} = 0$ indicating that there is no anomaly. Indeed the very geometrical feature associated with the product state of spins where all spins have there orientations in the same direction representing chiral spinors on a lattice does not allow chiral anomaly to exist. The vanishing of chiral anomaly on a lattice leads to the fermion doubling of chiral spinors on a lattice [30-32]. Thus we find that the relationship of the entanglement entropy with the Berry phase and consequently with chiral anomaly has its relevance in the fermion doubling problem on a lattice [33-34]. In fact the disentangled spin state with all spins polarized guarantees vanishing of chiral anomaly on a lattice which is the root cause of species doubling problem.

4. Discussion

In the present formulation we have considered that a fermion may be viewed as a scalar particle attached with a magnetic flux line. Indeed this formulation arises from the stochastic quantization of a fermion in the framework of Nelson's stochastic mechanics. In this framework a massive fermion appears as a soliton (skyrmion) and the mass arises from the inherent magnetic field [14]. It has been pointed out that this inherent magnetic flux line is responsible for the entanglement of spins in a spin system. It has been shown in an earlier paper [11] that the entanglement of two identical fermions is related to the entanglement of the two distinguishable spins.

5. Conclusion

It has been shown that the measure of entanglement in a mixed state given by concurrence is related to the Berry phase attained by a spin state when it evolves in a closed path [9-11]. It has been shown in a recent paper [15] that the Berry phase appears as the Pauli phase when two identical fermions are interchanged. In view of this we can view that entanglement is a consequence of Fermi statistics [17]. In this context it may be added that a polarized photon may be viewed as a chiral fermion and the entanglement between two polarizations of photons can be transcribed in the framework of entanglement of spins.

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